

# ON ENHANCED BINDING AND RELATED EFFECTS IN THE NON- AND SEMI-RELATIVISTIC PAULI-FIERZ MODELS

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**ABSTRACT.** We prove enhanced binding and increase of binding energies in the non- and semi-relativistic Pauli-Fierz models, for arbitrary values of the fine-structure constant and the ultra-violet cut-off, and discuss the resulting improvement of exponential localization of ground state eigenvectors. For the semi-relativistic model we also discuss the increase of the renormalized electron mass and determine the linear leading order term in the asymptotics of the self-energy, as the ultra-violet cut-off goes to infinity.

## 1. INTRODUCTION

A moving electron emits and absorbs electromagnetic radiation and is hence always accompanied by a cloud of so-called soft photons. Together with its photon cloud the electron behaves like a particle whose mass is larger than the bare mass of the electron. In an electrostatic potential it is thus easier to bind an electron interacting with the quantized photon field than the electron alone if the photon field were neglected. It is well-known that these phenomena may be described mathematically in the framework of non-relativistic (NR) quantum electrodynamics as follows:

First, we recall an effect called *enhanced binding* due to the quantized radiation field. We consider the non-relativistic electron Hamiltonian

$$(1.1) \quad h_{\text{nr}}(V) := -\frac{1}{2}\Delta_{\mathbf{x}} + V,$$

acting in the Hilbert space  $L^2(\mathbb{R}^3)$ . We suppose the potential  $V = V_+ - V_-$  to have a short range negative part and recall the following definition, which also applies to the semi-relativistic operator  $h_{\text{sr}}(V)$  introduced in (1.4) below:

**Definition 1.1.** Let  $0 \leq V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$  and let  $0 \leq V_- \not\equiv 0$  satisfy  $V_- \in L^{3/2}(\mathbb{R}^3)$ , if  $\sharp = \text{nr}$ , and  $V_- \in L^{3/2} \cap L^3(\mathbb{R}^3)$ , if  $\sharp = \text{sr}$ . Define  $h_{\sharp}(V_{\lambda})$  with  $V_{\lambda} := V_+ - \lambda V_-$ ,  $\lambda \geq 0$ , via a semi-bounded sum of quadratic forms. We say 1 is a coupling constant threshold for  $V_{\lambda}$ , iff

- (1)  $\inf \sigma_{\text{ess}}(h_{\sharp}(V_+)) = 0$ .
- (2)  $\inf \sigma(h_{\sharp}(V_{\lambda})) = 0$ , for  $\lambda \in (0, 1]$ , and  $\inf \sigma(h_{\sharp}(V_{\lambda})) < 0$ , for  $\lambda > 1$ .

The existence of coupling constant thresholds is a consequence of the variational principle and the famous Cwikel-Lieb-Rosenbljum bound,

$$\mathrm{Tr}[\mathbb{1}_{(-\infty,0)}(h_{\mathrm{nr}}(V_\lambda))] \leqslant \mathfrak{c} \lambda^{3/2} \int_{\mathbb{R}^3} V_-^{3/2}(\mathbf{x}) d^3\mathbf{x}.$$

Here  $\mathrm{Tr}$  denotes the trace and  $\mathbb{1}_M(T)$  denotes the spectral projection associated with some self-adjoint operator  $T$  and a Borel set  $M \subset \mathbb{R}$ . So the left hand side in the Cwikel-Lieb-Rosenbljum bound counts all negative eigenvalues of  $h_{\mathrm{nr}}(V_\lambda)$  including multiplicities and, thus, has to be zero, for sufficiently small  $\lambda > 0$  and  $V_- \in L^{3/2}(\mathbb{R}^3)$ . Changing  $V_- \not\equiv 0$  by a multiplicative constant, if necessary, we may thus achieve that 1 is a coupling constant threshold. Next, we take the interaction with the quantized radiation field into account and consider the NR Pauli-Fierz operator

$$\mathbb{H}_{\mathrm{nr}}(V) := \frac{1}{2}(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} + \mathfrak{e} \mathbb{A}))^2 + V + H_{\mathrm{f}},$$

where  $\boldsymbol{\sigma}$  is a vector containing the Pauli matrices,  $\mathbb{A} \equiv \mathbb{A}_\Lambda$  is the quantized vector potential in the Coulomb gauge with ultra-violet (UV) cutoff at  $\Lambda > 0$ ,  $H_{\mathrm{f}}$  is the radiation field energy, and  $\mathfrak{e} \in \mathbb{R}$  models the fine-structure constant. We set

$$\mathbb{E}_{\mathrm{nr}}(V) := \inf \sigma(\mathbb{H}_{\mathrm{nr}}(V)),$$

for any reasonable potential  $V$ . Back in our example  $V_- \in L^{3/2}(\mathbb{R}^3)$  we say that *enhanced binding occurs*, iff 1 is a coupling constant threshold for  $V_\lambda$  and  $\mathbb{E}_{\mathrm{nr}}(V_\lambda)$  is an eigenvalue of  $\mathbb{H}_{\mathrm{nr}}(V_\lambda)$ , for some  $\lambda < 1$ . Notice that, if 1 is a coupling constant threshold, then  $\inf \sigma(h_{\mathrm{nr}}(V_\lambda))$  cannot be an eigenvalue of  $h_{\mathrm{nr}}(V_\lambda)$ , for any  $\lambda < 1$ . In order to observe this effect it suffices to show that

$$(1.2) \quad \mathbb{E}_{\mathrm{nr}}(0) - \mathbb{E}_{\mathrm{nr}}(V_\lambda) > 0, \quad \lambda \geqslant 1 - \delta,$$

for some  $\delta > 0$ . In fact, according to [GLL] the inequality in (1.2) is a sufficient condition for  $\mathbb{E}_{\mathrm{nr}}(V_\lambda)$  to be an eigenvalue.

In the past decade many mathematical articles dealt with enhanced binding in NR quantum electrodynamics. In the earliest one [HiSp] the dipole approximation to the NR Pauli-Fierz model without spin is considered and enhanced binding is established, for all sufficiently large values of  $\mathfrak{e}$ . The previous works on the full NR Pauli-Fierz model [BLV, BeVu, CEH, CaHa, CVV, HVV] (with various additional conditions on  $V_-$  and sometimes without spin) provide *complete* proofs of enhanced binding under the assumptions that  $|\mathfrak{e}| > 0$  and/or  $\Lambda > 0$  be sufficiently small. Saying this we should, however, point out the article [CVV] where a general criterion for the occurrence of enhanced binding is established, namely existence of an eigenvalue at the bottom of the spectrum of the fiber Hamiltonian corresponding to total momentum  $\mathbf{0}$  of the translation invariant electron-photon system as well as equality of this eigenvalue and the self-energy of the electron. While the conclusion proved in [CVV] is always

applicable, no matter how big  $|\mathfrak{e}| > 0$  and  $\Lambda > 0$  are, the existence of that eigenvalue has been shown so far only for sufficiently small values of  $|\mathfrak{e}| > 0$  and/or  $\Lambda > 0$  [Ch, CFP].

The criterion established in [CVV] can also be applied to prove the *increase of binding energy* due to the quantized radiation field. Here one assumes that the electronic Hamiltonian with potential  $V$  *does* have discrete eigenvalues below zero, which is henceforth again assumed to be the minimum of its essential spectrum. Let  $|e_V|$  be the absolute value of the lowest (strictly negative) eigenvalue of  $h_{\text{nr}}(V)$ . By definition the *binding energy is increased*, iff

$$(1.3) \quad \mathbb{E}_{\text{nr}}(0) - \mathbb{E}_{\text{nr}}(V) > |e_V|.$$

In [BCVV, Ha] this effect is observed on the basis of asymptotic expansions as  $\mathfrak{e}$  goes to zero. The estimates obtained in these articles as well as the bounds on the shift of coupling constant thresholds in [BLV, CEH] come along with detailed quantitative information on the coefficients in the expansions. While these quantitative aspects are interesting in their own right, it has always been expected that there should exist entirely non-perturbative proofs without any smallness assumptions on  $\mathfrak{e}$  or  $\Lambda$  covering, in particular, the physical value  $\mathfrak{e}^2 \approx 1/137$  no matter how big  $\Lambda$  is chosen. The first main achievement of the present article fills this gap left open in the previous work. Namely, we prove enhanced binding and increase of binding energies in the NR Pauli-Fierz model, *for all values of  $|\mathfrak{e}| > 0$  and  $\Lambda > 0$* . To this end we employ variational arguments similar to those in [CVV] with the crucial difference, however, that our trial functions are constructed by means of *minimizing sequences* of fiber Hamiltonians (instead of eigenvectors) whose detailed properties are unknown a priori. The main new technical difficulty is to provide estimates holding uniformly along such minimizing sequences. These estimates show that actually no non-trivial a priori knowledge on the mass shell and no deep results on the existence of ground states of fiber Hamiltonians are required to give a qualitative discussion of enhanced binding and increased binding energies. In the discussion of enhanced binding we are also able to relax earlier assumptions on the short range potential whose (non-vanishing) negative part need not satisfy any other condition than being in  $L^{3/2}(\mathbb{R}^3)$ . The mildest condition on the local singularities stated in the quoted literature is  $V_- \in L^4_{\text{loc}}(\mathbb{R}^3)$  [BLV]. As a technical prerequisite some information on the convergence of electronic eigenfunctions to threshold energy states is needed here (see also [CVV]). Corresponding results are supplied by [SøSt] (also in the semi-relativistic case discussed below) for bounded and integrable potentials. We shall push these results a little forward to the broader class of potentials considered here.

The second purpose of our paper is to study the enhancement of binding and the increase of binding energies in the *semi*-relativistic (SR) Pauli-Fierz model, whose mathematical analysis has been initiated in [FGS, MiSp]. This model is

obtained by replacing the symbol  $\frac{1}{2}|\xi|^2$  of the kinetic energy in the NR model by its relativistic analog,  $\sqrt{|\xi|^2 + 1} - 1$ . Thus, the electron Hamiltonian reads

$$(1.4) \quad h_{\text{sr}}(V) := \sqrt{1 - \Delta_{\mathbf{x}}} - 1 + V,$$

and the full SR Pauli-Fierz Hamiltonian for the interacting system is still obtained via minimal coupling to the quantized radiation field,

$$(1.5) \quad \mathbb{H}_{\text{sr}}(V) := \sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} + \mathfrak{e} \mathbb{A}))^2 + 1} - 1 + V + H_{\text{f}}.$$

In the SR case it follows from [Cw, Da] that

$$\text{Tr}[\mathbb{1}_{(-\infty, 0)}(h_{\text{sr}}(V_{\lambda}))] \leq \mathfrak{c} \int_{\mathbb{R}^3} ((\lambda V_{-}(\mathbf{x}))^{3/2} + (\lambda V_{-}(\mathbf{x}))^3) d^3\mathbf{x}.$$

So, we again expect to observe an enhanced binding, for non-zero  $V_{-}$  belonging to  $L^{3/2}(\mathbb{R}^3)$  (this condition is due to  $\sqrt{|\xi|^2 + 1} - 1 \sim \frac{1}{2}|\xi|^2$  for small  $|\xi|$ ) as well as to  $L^3(\mathbb{R}^3)$  (due to  $\sqrt{|\xi|^2 + 1} - 1 \sim |\xi|$  for large  $|\xi|$ ). In fact, criteria for the existence of ground states for the relevant class of short range potentials are given in [KMS3]. A related problem is treated in [HiSa2] where  $N$  relativistic spin-less particles in a short range potential interacting via a linearly coupled bosonic field are considered. If a scaling parameter in front of the creation and annihilation operators is sufficiently large (weak coupling limit) and if the coupling constant in front of the interaction lies in a certain bounded interval, then the authors are able to show existence of a unique ground state of the total Hamiltonian. A non-strict inequality analogous to (1.3) has been obtained in the SR case first in [HiSa1].

As a byproduct of our analysis we verify that the renormalized electron mass in the SR Pauli-Fierz model is always strictly larger than the bare mass of the electron, as soon as it may be defined (as the inverse second radial derivative of the mass shell at zero). For small  $|\mathfrak{e}| > 0$ , depending on  $\Lambda$ , the *existence* of the renormalized electron mass in the SR Pauli-Fierz model has been proved recently by the present authors in [KöMa2]. In another application of our ideas we determine the linear leading order term in the asymptotics of the ground state energy of the free SR Pauli-Fierz operator, as  $\Lambda$  goes to infinity. Asymptotically linear upper and lower bounds on the self-energy have been obtained earlier in [LiLo1].

The organization of this article is given as follows. In Subsection 2.1 we introduce all operators studied here more precisely. All of our main results are stated precisely in Subsection 2.2. In Section 3 we develop the crucial technical estimates used to derive our main theorems. In Sections 4 and 5 we apply them to the NR and SR models, respectively. In the appendix we recall some Birman-Schwinger principles and extend some results from [SøSt] on the convergence of eigenfunctions to threshold energy states.

## 2. MODELS AND MAIN RESULTS

### 2.1. Definition of the models.

2.1.1. *Bosonic Fock space.* First, we fix some notation for operators acting in the state space of the photon field, the bosonic Fock space,  $\mathcal{F}_b$ . In what follows an italic  $k$  always denotes a tuple  $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$  and  $\mathcal{A}$  is a non-void open subset of  $\mathbb{R}^3$ . (In applications we encounter the examples  $\mathcal{A} = \mathbb{R}^3$  or  $\mathcal{A} = \{|\mathbf{k}| > m\}$  with  $m > 0$ .) For every  $n \in \mathbb{N}$ , let  $\mathcal{S}_n$  denote the orthogonal projection in  $L^2((\mathcal{A} \times \mathbb{Z}_2)^n)$  onto the space of permutation symmetric functions. That is,

$$(\mathcal{S}_n \psi^{(n)})(k_1, \dots, k_n) := \frac{1}{n!} \sum_{\pi} \psi^{(n)}(k_{\pi(1)}, \dots, k_{\pi(n)}),$$

almost everywhere, for  $\psi^{(n)} \in L^2((\mathcal{A} \times \mathbb{Z}_2)^n)$ , the sum running over all permutations of  $\{1, \dots, n\}$ . Then the bosonic Fock space modeled over the one photon Hilbert space  $\mathfrak{h} := L^2(\mathcal{A} \times \mathbb{Z}_2, dk)$ ,  $\int dk := \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathcal{A}} d^3 \mathbf{k}$ , is the direct sum

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \mathcal{F}_b^{(n)}, \quad \text{with } \mathcal{F}_b^{(0)} := \mathbb{C}, \quad \mathcal{F}_b^{(n)} := \mathcal{S}_n L^2((\mathcal{A} \times \mathbb{Z}_2)^n), \quad n \in \mathbb{N}.$$

The vector  $\Omega := \{1, 0, 0, \dots\} \in \mathcal{F}_b$  is called the vacuum. We denote by  $\mathcal{C}$  the dense subspace of all  $\{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b$  such that only finitely many  $\psi^{(n)}$  are non-zero and each  $\psi^{(n)}$ ,  $n \in \mathbb{N}$ , has a compact support.

For  $f \in \mathfrak{h}$ , let  $a^\dagger(f)$  and  $a(f)$  denote the standard bosonic creation and annihilation operators, respectively. Setting  $a^\dagger(f)^{(n)} \psi^{(n)} := (n+1)^{1/2} \mathcal{S}_{n+1}(f \otimes \psi^{(n)})$ , for  $\psi^{(n)} \in \mathcal{F}_b^{(n)}$ ,  $n \in \mathbb{N}_0$ , the creation operator is the closed operator given by  $a^\dagger(f) \psi := \{0, a^\dagger(f)^{(0)} \psi^{(0)}, a^\dagger(f)^{(1)} \psi^{(1)}, \dots\}$ , for all  $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b$  in its maximal domain, and  $a(f) := a^\dagger(f)^*$ . The following canonical commutation relations (CCR) are satisfied on a suitable dense domain (e.g., on  $\mathcal{C}$ ),

$$[a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0, \quad [a(f), a^\dagger(g)] = \langle f, g \rangle \mathbb{1},$$

for  $f, g \in \mathfrak{h}$ . The second quantization of a real-valued Borel function,  $\varkappa$ , on  $\mathcal{A}$ , is the self-adjoint operator in  $\mathcal{F}_b$  given by  $d\Gamma(\varkappa)|_{\mathcal{F}_b^{(0)}} := 0$  and

$$d\Gamma(\varkappa)|_{\mathcal{F}_b^{(n)}} \psi^{(n)}(k_1, \dots, k_n) := (\varkappa(\mathbf{k}_1) + \dots + \varkappa(\mathbf{k}_n)) \psi^{(n)}(k_1, \dots, k_n),$$

for  $n \in \mathbb{N}$ ,  $k_j = (\mathbf{k}_j, \lambda_j)$ . The multiplication operator  $d\Gamma(\varkappa)$  is defined on its maximal domain. For instance, the field energy operator,  $H_f$ , and the field momentum operator,  $\mathbf{p}_f$ , are defined by

$$H_f := d\Gamma(\omega), \quad \mathbf{p}_f := d\Gamma(\boldsymbol{\mu}) := (d\Gamma(\mu_1), d\Gamma(\mu_2), d\Gamma(\mu_3)).$$

The physically relevant choices of  $\omega$  and  $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$  are given in Example 2.2 below. Henceforth, we shall, however, only assume that  $\omega, \mu_1, \mu_2, \mu_3$  :

$\mathcal{A} \rightarrow \mathbb{R}$  are measurable such that

$$(2.1) \quad 0 < \omega(\mathbf{k}) \leq d(|\mathbf{k}| + 1), \quad |\boldsymbol{\mu}(\mathbf{k})| \leq d\omega(\mathbf{k}), \quad \text{a.e. } \mathbf{k} \in \mathcal{A},$$

for some  $d > 0$ . The following standard estimates shall be useful later on,

$$(2.2) \quad \begin{aligned} \|a(f_1) \dots a(f_n) \psi\| &\leq \|f_1\|_{1/2} \dots \|f_n\|_{1/2} \|H_{\mathbf{f}}^{n/2} \psi\|, \\ \|a^\dagger(f) \psi\|^2 &\leq \|f\|_{1/2}^2 \|H_{\mathbf{f}}^{1/2} \psi\|^2 + \|f\|^2 \|\psi\|^2, \end{aligned}$$

with  $\|f\|_{1/2} := \|\omega^{-1/2} f\|$ , for all  $f, f_j \in \mathfrak{h}$  and  $\psi \in \mathcal{F}_b$  such that the right hand sides are finite. For every  $f \in \mathfrak{h}$ , the operator  $2^{-1/2}(a^\dagger(f) + a(f))$  is essentially self-adjoint on  $\mathcal{C}$ . We denote its self-adjoint extension by  $\varphi(f)$  and write  $\varphi(\mathbf{f}) := (\varphi(f_1), \varphi(f_2), \varphi(f_3))$ , for a triple of photon wave functions  $\mathbf{f} = (f_1, f_2, f_3) \in \mathfrak{h}^3$ .

**2.1.2. Fiber Hamiltonians.** We next define Hamiltonians acting in  $\mathbb{C}^2 \otimes \mathcal{F}_b$  which is henceforth referred to as the fiber Hilbert space. For reasons illustrated by Example 2.3 below we work with general conditions on the dispersion relation  $\omega$ , the vector field  $\boldsymbol{\mu}$ , and the coupling function  $\mathbf{G}$ :

*Hypothesis 2.1.*  $\omega : \mathcal{A} \rightarrow \mathbb{R}$ ,  $\boldsymbol{\mu} : \mathcal{A} \rightarrow \mathbb{R}^3$ , and  $\mathbf{G} : \mathcal{A} \times \mathbb{Z}_2 \rightarrow \mathbb{R}^3$  are measurable and satisfy (2.1) and

$$(2.3) \quad \|\mathbf{G}\| \geq g, \quad \int \omega^\ell |\mathbf{G}|^2 \leq d^2, \quad \ell \in \{-1, 0, \dots, 13\},$$

respectively, for some  $d \geq 1$ ,  $g > 0$ . Moreover,

$$(2.4) \quad \|\mathbb{1}_{\{\omega \leq \delta\}} \mathbf{G}\| + \|\mathbb{1}_{\{\omega \leq \delta\}} \omega^{-1/2} \mathbf{G}\| \leq r(\delta), \quad \delta > 0,$$

for some non-negative function  $r : (0, \infty) \rightarrow \mathbb{R}$  with  $r(\delta) \rightarrow 0$ ,  $\delta \downarrow 0$ .  $\diamond$

The somewhat mysterious bound  $\ell \leq 13$  in (2.3) is due to the application of a certain higher order estimate in Lemma 3.3 below. The function  $r$  is introduced in order to treat several choices of  $\mathbf{G}$  at the same time and to quantify their infra-red behavior in a uniform fashion.

*Example 2.2.* (i) Physically relevant choices of  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  fulfilling Hypothesis 2.1 are given by  $\omega(\mathbf{k}) := |\mathbf{k}|$ ,  $\boldsymbol{\mu}(\mathbf{k}) := \mathbf{k}$ , for  $\mathbf{k} \in \mathcal{A} := \mathbb{R}^3$ , and

$$(2.5) \quad \mathbf{G} = \varrho(|\mathbf{k}|) \boldsymbol{\varepsilon}(\mathring{\mathbf{k}}, \lambda), \quad \mathring{\mathbf{k}} := \mathbf{k}/|\mathbf{k}|,$$

for almost every  $\mathbf{k}$  and  $\lambda \in \mathbb{Z}_2$ , where  $\varrho$  is some measurable real function with

$$0 < \int_0^\infty (t + t^{15}) \varrho^2(t) dt < \infty,$$

and  $\{\mathring{\mathbf{k}}, \boldsymbol{\varepsilon}(\mathring{\mathbf{k}}, 0), \boldsymbol{\varepsilon}(\mathring{\mathbf{k}}, 1)\}$  is an oriented orthonormal basis of  $\mathbb{R}^3$ , for a.e.  $\mathring{\mathbf{k}} \in S^2$ .

(ii) A common special case of (i) is given by  $\mathbf{G} := \mathbf{G}_\Lambda^\epsilon$  with

$$(2.6) \quad \mathbf{G}_\Lambda^\epsilon(\mathbf{k}, \lambda) := (2\pi)^{-3/2} \epsilon |\mathbf{k}|^{-1/2} \mathbb{1}_{|\mathbf{k}| < \Lambda} \epsilon(\mathbf{k}, \lambda),$$

where  $\epsilon \in \mathbb{R} \setminus \{0\}$  and  $\Lambda > 0$  is an UV cutoff parameter.  $\diamond$

The main reason why we introduce the quantities  $d$ ,  $g$ , and  $r$  in the above hypothesis is the following example. The modified versions of the physical choices of  $\omega$ ,  $\boldsymbol{\mu}$ , and  $\mathbf{G}$  defined there appear in proofs of the existence of ground states; see, for instance, [KMS1] and the proof of Corollary 2.7 below.

*Example 2.3.* Let  $\omega$ ,  $\boldsymbol{\mu}$ , and  $\mathbf{G}$  be as in Example 2.2(ii).

(i) Let  $\mathbf{G}_m := \mathbb{1}_{\{\omega \geq m\}} \mathbf{G}_\Lambda^\epsilon$  and  $m_0 \in (0, \Lambda)$ . Trivially, all  $(\omega, \boldsymbol{\mu}, \mathbf{G}_m)$  with  $0 < m \leq m_0$  fulfill Hypothesis 2.1 with the same suitable choices of  $d$ ,  $g$ ,  $r$ .

(ii) Pick some  $m > 0$  and replace  $\mathbb{R}^3$  by  $\mathcal{A}_m := \{|\mathbf{k}| > m\}$  in Example 2.2. Set  $Q(\boldsymbol{\nu}) := \boldsymbol{\nu} + (-1/2, 1/2]^3$  and  $Q_\epsilon(\boldsymbol{\nu}) := (\epsilon Q(\boldsymbol{\nu})) \cap \mathcal{A}_m$ , for all  $\epsilon > 0$  and  $\boldsymbol{\nu} \in \mathbb{Z}^3$ . Set  $\omega_\epsilon|_{Q_\epsilon(\boldsymbol{\nu})} := \inf_{Q_\epsilon(\boldsymbol{\nu})} \omega$ , let  $\boldsymbol{\mu}_\epsilon|_{Q_\epsilon(\boldsymbol{\nu})}$  be constantly equal to some arbitrary vector in  $\overline{Q_\epsilon(\boldsymbol{\nu})}$ , and let  $\mathbf{G}_\epsilon|_{Q_\epsilon(\boldsymbol{\nu})}$  be constantly equal to the average of  $\mathbf{G}_\Lambda^\epsilon$  over  $Q_\epsilon(\boldsymbol{\nu})$ . Then we find ( $m$ -dependent)  $\epsilon_0 > 0$ ,  $d$ ,  $g$ , and  $r$  such that all  $(\omega_\epsilon, \boldsymbol{\mu}_\epsilon, \mathbf{G}_\epsilon)$ ,  $0 < \epsilon < \epsilon_0$ , fulfill Hypothesis 2.1 with these fixed choices of  $d$ ,  $g$ , and  $r$ .  $\diamond$

Let  $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$  be the triple of Pauli spin matrices and write  $\boldsymbol{\sigma} \cdot \mathbf{v} := \sigma_1 v_1 + \sigma_2 v_2 + \sigma_3 v_3$ , for a vector  $\mathbf{v} = (v_1, v_2, v_3)$  whose entries are complex numbers or suitable operators. For every  $\mathbf{p} \in \mathbb{R}^3$ , we then define

$$\mathbf{v}(\mathbf{p}) := \mathbf{p} - \mathbf{p}_f + \varphi(\mathbf{G}), \quad w(\mathbf{p}) := \boldsymbol{\sigma} \cdot \mathbf{v}(\mathbf{p}).$$

Applying Nelson's commutator theorem with test operator  $H_f + 1$  we verify that  $w(\mathbf{p})$  is essentially self-adjoint on any core of  $H_f$ . We denote its self-adjoint closure starting from  $\mathcal{C}$  again by the same symbol and define

$$\hat{\tau}_{\text{nr}}(\mathbf{p}) := \frac{1}{2} w(\mathbf{p})^2, \quad \hat{\tau}_{\text{sr}}(\mathbf{p}) := \sqrt{w(\mathbf{p})^2 + 1} - 1,$$

by means of the spectral calculus. Next, we define fiber Hamiltonians

$$H_\sharp(\mathbf{p}) := \hat{\tau}_\sharp(\mathbf{p}) + H_f, \quad \sharp \in \{\text{nr}, \text{sr}\},$$

as Friedrichs extensions starting from  $\mathcal{C}$ . For  $\mathbf{G} = \mathbf{0}$ , we denote them by

$$H_{\text{nr}}^0(\mathbf{p}) := \frac{1}{2} (\mathbf{p} - \mathbf{p}_f)^2 + H_f, \quad H_{\text{sr}}^0(\mathbf{p}) := \sqrt{(\mathbf{p} - \mathbf{p}_f)^2 + 1} - 1 + H_f.$$

It is known [KöMa2, Lemma 2.2(ii)] that  $\mathcal{D}(H_{\text{sr}}(\mathbf{p})) = \mathcal{D}(H_f)$  and, for all  $\epsilon > 0$ ,

$$(2.7) \quad \|(H_{\text{sr}}(\mathbf{p}) - H_{\text{sr}}^0(\mathbf{p})) \varphi\| \leq \epsilon \|H_{\text{sr}}^0(\mathbf{p}) \varphi\| + \mathfrak{c}(\epsilon, d) \|\varphi\|, \quad \varphi \in \mathcal{D}(H_f).$$

In particular,  $\mathcal{C}$  is a core for  $H_{\text{sr}}(\mathbf{p})$ . The mass shells are defined by

$$E_\sharp(\mathbf{p}) := \inf \sigma(H_\sharp(\mathbf{p})), \quad \mathbf{p} \in \mathbb{R}^3.$$

2.1.3. *Total Hamiltonians.* Finally, we introduce the Hamiltonians generating the time evolution of the combined electron-photon system. The total Hilbert space is

$$\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_b = \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathcal{F}_b d^3 \mathbf{x}.$$

The quantized vector potential,  $\mathbb{A}$ , is the triple of operators given by

$$\mathbb{A} := \int_{\mathbb{R}^3}^{\oplus} \mathbb{1}_{\mathbb{C}^2} \otimes \varphi(e^{-i\boldsymbol{\mu} \cdot \mathbf{x}} \mathbf{G}) d^3 \mathbf{x}.$$

We drop all trivial tensor factors in the notation in what follows ( $-i\nabla_{\mathbf{x}} \equiv -i\nabla_{\mathbf{x}} \otimes \mathbb{1}$ ,  $H_f \equiv \mathbb{1} \otimes H_f$ , etc.) and define

$$\mathbb{W} := \boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} + \mathfrak{e} \mathbb{A})$$

on the domain

$$\mathcal{D}_1 := \mathcal{D}(-\Delta_{\mathbf{x}}) \cap \mathcal{D}(H_f)$$

to begin with. An application of Nelson's commutator theorem shows that  $\mathbb{W}$  is essentially self-adjoint on any core of  $-\Delta_{\mathbf{x}} + H_f$  and in particular on  $\mathcal{D}_1$  and on the algebraic tensor product

$$\mathcal{D} := C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{C}.$$

Denoting the closure of  $\mathbb{W}$  again by the same symbol we define

$$(2.8) \quad \tau_{\text{nr}} := \frac{1}{2} \mathbb{W}^2, \quad \tau_{\text{sr}} := \sqrt{\mathbb{W}^2 + 1} - 1,$$

by the spectral calculus.

Next, let  $\sharp \in \{\text{nr}, \text{sr}\}$ , recall the notation (1.1) and (1.4), and assume that  $V \in L_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R})$  satisfies

$$(2.9) \quad c_V := -\inf \{ \langle \psi, h_{\sharp}(V) \psi \rangle : \psi \in C_0^\infty(\mathbb{R}^3), \|\psi\| = 1 \} < \infty.$$

Then it is known that

$$(2.10) \quad \mathbb{H}_{\sharp}(V) := \tau_{\sharp} + V + H_f \geqslant -c_V - \mathfrak{c} d^2$$

in the sense of quadratic forms on  $\mathcal{D}$ , for some universal constant  $\mathfrak{c} > 0$ . In the NR case this is a well-known consequence of diamagnetic inequalities (see, e.g., the review in [KMS3]) and relative bounds on the magnetic field with respect to  $H_f$ . In the SR case (2.10) follows from [KMS3, Theorem 3.4]. Therefore, the quadratic forms of  $h_{\sharp}(V)$  and  $\mathbb{H}_{\sharp}(V)$  equipped with the domains  $C_0^\infty(\mathbb{R}^3)$  and  $\mathcal{D}$ , respectively, are closable. We denote the self-adjoint operators representing the closures of these forms again by the same symbols. According to [Hi1] (in the NR case) and [KMS2] (in the SR case) the operators  $\mathbb{H}_{\sharp}(0)$  are essentially self-adjoint on  $\mathcal{D}$ . The domain of  $\mathbb{H}_{\text{sr}}(0)$  is  $\mathcal{D}((-\Delta_{\mathbf{x}})^{1/2} + H_f)$  [KMS2]. We set

$$\mathbb{E}_{\sharp}(V) := \inf \sigma(\mathbb{H}_{\sharp}(V)).$$



2.1.4. *Fiber decompositions.* Setting

$$(2.11) \quad U_{\mathbf{q}} := e^{-i(\mathbf{q}-\mathbf{p}_f) \cdot \mathbf{x}}$$

and denoting the (partial) Fourier transform with respect to  $\mathbf{x}$  by  $\mathcal{F}$ , we observe that  $U_{\mathbf{q}}^* \mathcal{F}^* \int_{\mathbb{R}^3}^{\oplus} w(\mathbf{q} + \mathbf{p}) d^3 \mathbf{p} \mathcal{F} U_{\mathbf{q}}$  is a self-adjoint extension of  $w|_{\mathcal{D}}$  and, hence, equal to  $w$ . Using [ReSi, Theorem XIII.85] in the second step we conclude that

$$f(w) = U_{\mathbf{q}}^* \mathcal{F}^* f \left( \int_{\mathbb{R}^3}^{\oplus} w(\mathbf{q} + \mathbf{p}) d^3 \mathbf{p} \right) \mathcal{F} U_{\mathbf{q}} = U_{\mathbf{q}}^* \mathcal{F}^* \int_{\mathbb{R}^3}^{\oplus} f(w(\mathbf{q} + \mathbf{p})) d^3 \mathbf{p} \mathcal{F} U_{\mathbf{q}},$$

where  $f$  is  $x^2$  or a bounded Borel function. Writing  $f(x) = (x^2 + 1)^{1/2}$  as  $(x^2 + 1)(x^2 + 1)^{-1/2}$  we see that this choice of  $f$  is allowed, too. But then it follows that  $U_{\mathbf{q}}^* \mathcal{F}^* \int_{\mathbb{R}^3}^{\oplus} H_{\sharp}(\mathbf{q} + \mathbf{p}) d^3 \mathbf{p} \mathcal{F} U_{\mathbf{q}}$  is a self-adjoint extension of the essentially self-adjoint operator  $\mathbb{H}_{\sharp}(0)|_{\mathcal{D}}$ , whence we have the fiber decomposition

$$(2.12) \quad \mathcal{F} U_{\mathbf{q}} \mathbb{H}_{\sharp}(0) U_{\mathbf{q}}^* \mathcal{F}^* = \int_{\mathbb{R}^3}^{\oplus} H_{\sharp}(\mathbf{q} + \mathbf{p}) d^3 \mathbf{p}.$$

**Lemma 2.4.** *If  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  fulfill Hypothesis 2.1,  $\sharp \in \{\text{nr}, \text{sr}\}$ , and  $\mathbf{p} > 0$ , then there exist  $\epsilon_0 \equiv \epsilon_0(\mathbf{p}, d)$  and  $\mathbf{p}_* \equiv \mathbf{p}_*(d) > 0$ , such that*

$$(2.13) \quad \sup_{|\mathbf{p}| \leq \mathbf{p}} E_{\sharp} \leq \epsilon_0, \quad \mathbb{E}_{\sharp}(0) = \text{ess inf}_{|\mathbf{p}| \leq \mathbf{p}_*} E_{\sharp}(\mathbf{p}).$$

*Proof.* On account of (2.12) and [ReSi, Theorem XIII.85] we have  $\mathbb{E}_{\sharp}(0) = \text{ess inf}_{\mathbb{R}^3} E_{\sharp}$ . In the SR case the bounds  $\frac{1}{2}|\mathbf{p}| - \mathbf{c}(d) \leq E_{\text{sr}}(\mathbf{p}) \leq \frac{3}{2}|\mathbf{p}| + \mathbf{c}(d)$ , follow in a straightforward fashion from (2.7) and imply (2.13). Concerning the NR case, the upper bound  $E_{\text{nr}}(\mathbf{p}) \leq \mathbf{p}^2/2 + d^2$  follows immediately by testing with vectors in the vacuum sector. Finally, for  $\delta \in (0, 1)$  sufficiently close to 1, we have, as quadratic forms on  $\mathcal{C}$ ,

$$\begin{aligned} H_{\text{nr}}(\mathbf{p}) &\geq (1 - \delta) \frac{1}{2}(\mathbf{p} - \mathbf{p}_f)^2 + (1 - \delta^{-1}) \frac{1}{2} \varphi(\mathbf{G})^2 + H_f \\ &\geq (1 - \delta) \frac{1}{2}(\mathbf{p} - \mathbf{p}_f)^2 + \mathbf{c}(1 - \delta^{-1}) d^2 (H_f + 1) + H_f \\ &\geq \mathbf{c}(d)^{-1} H_{\text{nr}}^0(\mathbf{p}) - \mathbf{c}'(d). \end{aligned}$$

But (2.1) implies  $d H_{\text{nr}}^0(\mathbf{p}) \geq \mathbb{1}_{|\mathbf{p}| < 1} \mathbf{p}^2/2 + \mathbb{1}_{|\mathbf{p}| \geq 1} (|\mathbf{p}| - 1/2)$  and we again obtain (2.13).  $\square$

The bounds in the previous proof are by no means optimal. Moreover, one can always show continuity of the mass shells and under physically reasonable assumptions they are supposed to attain their unique minimum at  $\mathbf{0}$ . We gave a very simple, self-contained proof since more detailed information than in Lemma 2.4 would not lead to any relevant simplifications in our proofs.

**2.2. Main results.** The first main results of the present paper are summarized in the following theorem. As already stressed above, what is crucial here is that Theorem 2.5 applies to the physical Example 2.2(ii) without any restrictions on  $|\mathfrak{e}|, \Lambda > 0$ . In the SR case the implications of Theorem 2.5 are new also when  $|\mathfrak{e}|, \Lambda > 0$  are small in that example. Recall the definitions of  $h_{\sharp}(V)$ ,  $\mathbb{H}_{\sharp}(V)$ , and  $\mathbb{E}_{\sharp}(V)$  in Sub-subsection 2.1.3.

**Theorem 2.5 (Increased and enhanced binding).** *Assume that  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  fulfill Hypothesis 2.1 with parameters  $d, g$ , and  $r$ . In the case  $\sharp = \text{nr}$  assume in addition that  $\omega, \boldsymbol{\mu}$ , and  $\mathbf{G}$  are as in Example 2.2(i).*

(a) *Let  $V \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R})$  satisfy (2.9). If*

$$(2.14) \quad \inf \sigma_{\text{ess}}(h_{\sharp}(V)) = 0, \quad e_{\sharp}(V) := \inf \sigma(h_{\sharp}(V)) < 0,$$

*then we find some  $c \equiv c(d, g, r, V) > 0$  such that*

$$\mathbb{E}_{\sharp}(0) - \mathbb{E}_{\sharp}(V) - e_{\sharp}(V) \geq c.$$

(b) *If  $\sharp = \text{nr}$ , let  $0 \leq V_- \in L^{3/2}(\mathbb{R}^3)$ ,  $V_- \not\equiv 0$ , and  $0 \leq V_+ = V_{+,1} + V_{+,2}$  with  $V_{+,1} \in L^{3/2}(\mathbb{R}^3)$  and  $V_{+,2} \in L^{\infty}(\mathbb{R}^3)$ . If  $\sharp = \text{sr}$ , let  $0 \leq V_- \in L^{3/2} \cap L^3(\mathbb{R}^3)$ ,  $V_- \not\equiv 0$ , and  $0 \leq V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$ . Set  $V_{\mu} := V_+ - \mu V_-$ ,  $\mu > 0$ , and assume that 1 is a coupling constant threshold for  $V_{\mu}$ . Then there exist  $c, \delta > 0$ , both depending only on  $d, g, r$ , and  $V_{\pm}$ , such that*

$$\mathbb{E}_{\sharp}(0) - \mathbb{E}_{\sharp}(V_{\mu}) \geq c, \quad \mu \geq 1 - \delta.$$

*Proof.* The proofs of this theorem in the cases  $\sharp = \text{nr}$  and  $\sharp = \text{sr}$  are given in Sections 4 and 5, respectively.  $\square$

In the NR case we restrict ourselves to the situation of Example 2.2(i) since we exploit the rotation invariance of the free Hamiltonian in that case.

*Example 2.6.* In the situation of Example 2.3(i) (resp. (ii) with fixed  $m > 0$ ) label all quantities defined by means of  $(\omega, \boldsymbol{\mu}, \mathbf{G}_m)$  (resp.  $(\omega_{\varepsilon}, \boldsymbol{\mu}_{\varepsilon}, \mathbf{G}_{\varepsilon})$ ) by a superscript  $m$  (resp.  $\varepsilon$ ). If  $V$  is as in Theorem 2.5(a) or  $V = V_+ - \mu V_-$ ,  $\mu \geq 1 - \delta$ , with  $V_{\pm}$  and  $\delta$  as in Theorem 2.5(b), then we find  $m_0, \varepsilon_0 > 0$  with

$$(2.15) \quad \inf_{0 < m \leq m_0} (\mathbb{E}_{\text{sr}}^m(0) - \mathbb{E}_{\text{sr}}^m(V)) > 0, \quad \inf_{0 < \varepsilon \leq \varepsilon_0} (\mathbb{E}_{\text{sr}}^{\varepsilon}(0) - \mathbb{E}_{\text{sr}}^{\varepsilon}(V)) > 0. \quad \diamond$$

**Corollary 2.7 (Existence of ground states).** *Assume we are in the situation of Example 2.2(ii) with arbitrary values of  $|\mathfrak{e}|, \Lambda > 0$ .*

(a) *In the case  $\sharp = \text{nr}$ , let  $V \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R})$  be infinitesimally form-bounded with respect to  $-\Delta_{\mathbf{x}}$ . In the case  $\sharp = \text{sr}$ , let  $V \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R})$  be relatively form-bounded with respect to  $(-\Delta_{\mathbf{x}})^{1/2}$  with relative form bound  $< 1$ . If (2.14) holds and  $V(\mathbf{x}) \rightarrow 0$ , as  $|\mathbf{x}| \rightarrow \infty$ , then  $\mathbb{H}_{\text{nr}}(V)$  (resp.  $\mathbb{H}_{\text{sr}}(V)$ ) has normalizable ground state eigenvectors.*

(b) If the potentials  $V_{\pm}$  are as in Theorem 2.5(b) with  $V_{\pm}(\mathbf{x}) \rightarrow 0$ ,  $|\mathbf{x}| \rightarrow \infty$ , then we find some  $\delta > 0$  such that  $\mathbb{H}_{\text{nr}}(V_{\mu})$  (resp.  $\mathbb{H}_{\text{sr}}(V_{\mu})$ ) has normalizable ground state eigenvectors, for every  $\mu \geq 1 - \delta$ .

*Proof.* In the NR the case both (a) and (b) follow from [GLL, Theorem 2.1] according to which binding, i.e. the inequality  $\mathbb{E}_{\text{nr}}(V) < \mathbb{E}_{\text{nr}}(0)$ , implies the existence of ground states. In the SR case (a) and (b) can be proved by straightforward modifications of the arguments in [KMS1] where the Coulomb potential is treated. The details are worked out in [KMS3]. In fact, according to (2.15) the uniform binding conditions postulated in Hypothesis 6.6 of [KMS3] are fulfilled and, hence, (a) and (b) are special cases of [KMS3, Theorem 8.1].  $\square$

It is possible to prove the existence of ground states of  $\mathbb{H}_{\text{sr}}(V_{\gamma})$  with  $V_{\gamma}(\mathbf{x}) := -\gamma/|\mathbf{x}|$  also in the critical case  $\gamma = 2/\pi$  where the relative form bound of  $V_{2/\pi}$  with respect to  $(-\Delta_{\mathbf{x}})^{1/2}$  is equal to one [KöMa1]. (For  $\gamma > 2/\pi$ , the quadratic form of  $\mathbb{H}_{\text{sr}}(V_{\gamma})$  is unbounded below [KMS1].) Note that Theorem 2.5(a) applies to  $\mathbb{H}_{\text{sr}}(V_{2/\pi})$ . It turns out that the decay rate of the spatial  $L^2$ -exponential localization of ground state eigenvectors of  $\mathbb{H}_{\sharp}(V)$  is strictly bigger than the decay rate of the electronic eigenfunctions (if any).

**Corollary 2.8 (Increase of localization).** *Assume we are in the situation of Example 2.2(ii) with arbitrary  $|\mathfrak{e}|, \Lambda > 0$ . Let  $V$  satisfy the conditions of Corollary 2.7(a) or (b) or suppose  $V(\mathbf{x}) = -(2/\pi)/|\mathbf{x}|$ . Let  $\Phi_{\sharp}$  be a ground state eigenvector of  $\mathbb{H}_{\sharp}(V)$ . Then, in the NR case,*

$$(2.16) \quad \forall \beta > 0 : \quad \beta^2/2 < \mathbb{E}_{\text{nr}}(0) - \mathbb{E}_{\text{nr}}(V) \Rightarrow e^{\beta|\mathbf{x}|} \Phi_{\text{nr}} \in \mathcal{H},$$

and in the SR case

$$(2.17) \quad \forall \beta \in (0, 1) : \quad 1 - (1 - \beta^2)^{1/2} < \mathbb{E}_{\text{sr}}(0) - \mathbb{E}_{\text{sr}}(V) \Rightarrow e^{\beta|\mathbf{x}|} \Phi_{\text{sr}} \in \mathcal{H}.$$

*Proof.* The bound (2.16) follows from [GLL]; see also [Gr]. The bound (2.17) is a special case of [KMS3, Theorem 5.1] where the decay rates found in an earlier localization estimate [MaSt] are improved. The crucial observation that led to the decay rates in (2.17) has been made in [KöMa1]. (Only the Coulomb potential is treated explicitly in [KöMa1, MaSt]; extensions to other potentials are, however, straightforward.)  $\square$

What is crucial about the previous corollary is the range of decay rates  $\beta$  allowed for in (2.16) and (2.17). For instance, suppose  $V$  satisfies the conditions of Theorem 2.5(a) in the SR case. Suppose further that  $|e_{\text{sr}}(V)| < 1$ , which will hold true for weak potentials  $V$  and is known to be true in the Coulomb case,  $V = V_{\gamma}$ , as long as the model is stable, i.e.  $\gamma \in (0, 2/\pi]$ ; see [RRSMS]. Then the exponential decay rate for ground state eigenfunctions of  $h_{\text{sr}}(V)$  is equal to  $\beta_{\text{el}} := (1 - (1 + e_{\text{sr}}(V))^2)^{1/2}$ ; see [CMS]. Since  $\mathbb{E}_{\text{sr}}(0) - \mathbb{E}_{\text{sr}}(V) > |e_{\text{sr}}(V)|$  by Theorem 2.5, there exist  $\beta \in (\beta_{\text{el}}, 1)$  such that  $e^{\beta|\mathbf{x}|} \Phi_{\text{sr}} \in \mathcal{H}$ .

In the SR case the bounds of Theorem 2.5 are consequences of certain bounds on the *fiber* Hamiltonians giving rise to some further interesting results. In order to state the first one we introduce the functions

$$(2.18) \quad T_{\text{sr}}(\mathbf{p}) := (\mathbf{p}^2 + 1)^{1/2} - 1, \quad S(\mathbf{p}) := 1 - (\mathbf{p}^2 + 1)^{-1/2}, \quad \mathbf{p} \in \mathbb{R}^3.$$

**Theorem 2.9 (Upper bound on the mass shell).** *Fix  $d, g$ , and  $r$  in Hypothesis 2.1 and let  $\mathbf{p}_* \in \mathbb{R}^3$ . Then we find some  $\gamma(\mathbf{p}_*) \equiv \gamma(\mathbf{p}_*, d, g, r) \in (0, 1)$  such that, for all  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  fulfilling Hypothesis 2.1 and  $\mathbf{p} \in \mathbb{R}^3$ ,*

$$(2.19) \quad \frac{1}{2}(E_{\text{sr}}(\mathbf{p}_* + \mathbf{p}) + E_{\text{sr}}(\mathbf{p}_* - \mathbf{p})) \leq T_{\text{sr}}(\mathbf{p}) + E_{\text{sr}}(\mathbf{p}_*) - \gamma(\mathbf{p}_*) S(\mathbf{p}).$$

*Proof.* The assertion is proved in Lemma 5.1(b).  $\square$

The previous theorem has the following immediate corollary according to which the renormalized electron mass (i.e. the inverse of  $(d^2/dt^2)E_{\text{sr}}(t\mathbf{u})|_{t=0}$ ) is always strictly larger than its bare mass, which equals 1 in the units chosen in this paper. The regularity assumptions on  $E_{\text{sr}}$  in the statement can be verified, at least for small coupling constants  $|\mathfrak{e}| > 0$  depending on  $\Lambda$ ; see [KöMa2, Theorem 7.1]. In this situation it is also known [KöMa2] that  $E_{\text{sr}}(\mathbf{0}) = \inf_{\mathbb{R}^3} E_{\text{sr}}$ .

**Corollary 2.10 (Renormalized electron mass).** *In the situation of Example 2.2(ii) let  $|\mathfrak{e}|, \Lambda > 0$  and let  $\gamma(\mathbf{0})$  be as in (2.19). If  $E_{\text{sr}}$  is twice continuously differentiable in a neighborhood of zero, then*

$$(2.20) \quad \frac{d^2}{dt^2} E_{\text{sr}}(t\mathbf{u})|_{t=0} \leq 1 - \gamma(\mathbf{0}) < 1, \quad \mathbf{u} \in \mathbb{R}^3, |\mathbf{u}| = 1.$$

*Proof.* By (2.19),  $\frac{1}{t^2}(E_{\text{sr}}(t\mathbf{u}) + E_{\text{sr}}(-t\mathbf{u}) - 2E_{\text{sr}}(t\mathbf{u})) \leq \frac{2}{t^2}\{(t^2\mathbf{u}^2 + 1)^{1/2} - 1 - \gamma(\mathbf{0})S(t\mathbf{u})\}$ ,  $t > 0$ . In the limit  $t \downarrow 0$  this gives (2.20), if  $E_{\text{sr}}$  is  $C^2$  near  $\mathbf{0}$ .  $\square$

As a final application we discuss the ultra-violet behavior of the ground state energy  $\mathbb{E}_{\text{sr}}(0)$ . In the rest of this section we only consider the situation of Example 2.2(ii). To state and prove our corresponding results we introduce the bare mass of the electron,  $m \geq 0$ , as an additional parameter and display the UV cutoff parameter  $\Lambda > 0$  explicitly in the notation. More precisely, if we choose  $\mathbf{G} = \mathbf{G}_\Lambda^\mathfrak{e}$  as in (2.6), then we denote  $w(\mathbf{p})$  and  $\mathfrak{w}$  as  $w_\Lambda(\mathbf{p})$  and  $\mathfrak{w}_\Lambda$ . For all  $m \geq 0$ ,  $\Lambda > 0$ , and  $\mathbf{p} \in \mathbb{R}^3$ , we then define

$$\begin{aligned} H_{\text{sr},\Lambda,m}(\mathbf{p}) &:= \sqrt{w_\Lambda(\mathbf{p})^2 + m^2} - m + H_{\mathfrak{f}} \quad \text{with domain } \mathcal{D}(H_{\mathfrak{f}}), \\ E_{\text{sr},\Lambda,m}(\mathbf{p}) &:= \inf \sigma(H_{\text{sr},\Lambda,m}(\mathbf{p})), \\ \mathbb{H}_{\text{sr},\Lambda,m}(0) &:= \sqrt{\mathfrak{w}_\Lambda^2 + m^2} - m + H_{\mathfrak{f}} \quad \text{with domain } \mathcal{D}((-\Delta_{\mathbf{x}})^{1/2} + H_{\mathfrak{f}}), \\ \mathbb{E}_{\text{sr},\Lambda,m} &:= \inf \sigma(\mathbb{H}_{\text{sr},\Lambda,m}(0)), \end{aligned}$$

so that

$$(2.21) \quad \mathbb{E}_{\text{sr},\Lambda,m} = \text{ess inf}_{\mathbf{p} \in \mathbb{R}^3} E_{\text{sr},\Lambda,m}(\mathbf{p}), \quad m \geq 0, \Lambda > 0.$$

On account of

$$0 \leq (t^2 + m_1^2)^{1/2} - m_1 - (t^2 + m_2^2)^{1/2} + m_2 \leq m_2 - m_1, \quad 0 \leq m_1 \leq m_2,$$

the difference of two Hamiltonians with different bare masses extends to a bounded operator on the whole Hilbert space with  $\|H_{\text{sr},\Lambda,m_1}(\mathbf{p}) - H_{\text{sr},\Lambda,m_2}(\mathbf{p})\| \leq |m_1 - m_2|$  and similarly for  $\mathbb{H}_{\text{sr},\Lambda,m}(0)$ . In particular, all remarks on the (essential) self-adjointness of the Hamiltonians with  $m = 1$  are actually valid, for all  $m \geq 0$ . Furthermore,

$$(2.22) \quad 0 \leq \left\{ \begin{array}{c} E_{\text{sr},\Lambda,m_1}(\mathbf{p}) - E_{\text{sr},\Lambda,m_2}(\mathbf{p}) \\ \mathbb{E}_{\text{sr},\Lambda,m_1} - \mathbb{E}_{\text{sr},\Lambda,m_2} \end{array} \right\} \leq m_2 - m_1, \quad 0 \leq m_1 \leq m_2.$$

In fact, every mass  $m > 0$  is related to the bare mass one by scaling: Let  $(u\psi)(\mathbf{k}, \lambda) = \Lambda^{3/2} \psi(\Lambda \mathbf{k}, \lambda)$  be the dilatation on  $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ , and let  $\Gamma(u)$  be the associated dilatation on the Fock space. The action of the unitary  $\Gamma(u)$  is characterized by the formulas

$$\begin{aligned} \Gamma(u) a(f) &= a(u f) \Gamma(u), \quad \Gamma(u) a^\dagger(f) = a^\dagger(u f) \Gamma(u), \quad \Gamma(u) H_f = \Lambda H_f \Gamma(u), \\ \Gamma(u) \mathbf{p}_f &= \Lambda \mathbf{p}_f \Gamma(u), \quad \Gamma(u) \Omega = \Omega. \end{aligned}$$

Moreover,  $u \mathbf{G}_\Lambda^\epsilon = \Lambda \mathbf{G}_1^\epsilon$  by (2.6). From these formulas we readily infer that

$$(2.23) \quad \Gamma(u) H_{\text{sr},\Lambda,m}(\mathbf{p}) = \Lambda H_{\text{sr},1,m/\Lambda}(\mathbf{p}/\Lambda) \Gamma(u).$$

In view of (2.21) and (2.22) this permits to get

$$(2.24) \quad \Lambda^{-1} \mathbb{E}_{\text{sr},\Lambda,1} = \mathbb{E}_{\text{sr},1,1/\Lambda} \uparrow \mathbb{E}_{\text{sr},1,0}, \quad \Lambda \uparrow \infty.$$

Our results imply that the limit in (2.24) is actually *non-zero*.

**Theorem 2.11 (UV-Asymptotics).** *In the situation of Example 2.2(ii) and with the notation introduced above we have  $\mathbb{E}_{\text{sr},1,0} > 0$ . In particular, the leading asymptotics of  $\mathbb{E}_{\text{sr},\Lambda,1}$  is linear in  $\Lambda \rightarrow \infty$ .*

An asymptotically linear growth of the self-energy has been observed earlier in [LiLo1]. Notice that the self-energy grows at least as fast as  $\mathfrak{c} \Lambda^{3/2}$  in the NR model; see [LiLo1].

In view of the above simple remarks the existence of a non-vanishing linear contribution to  $\mathbb{E}_{\text{sr},\Lambda,1}$  is an immediate consequence of the results of Section 3:

*Proof.* Applying successively (2.21), (2.22), and (2.13) we get

$$\mathbb{E}_{\text{sr},1,0} = \text{ess inf}_{\mathbb{R}^3} E_{\text{sr},1,0} \geq \text{ess inf}_{\mathbb{R}^3} E_{\text{sr},1,1} = \text{ess inf}_{|\mathbf{p}| \leq \mathbf{p}_*} E_{\text{sr},1,1}(\mathbf{p}).$$

By (3.7) below the last essential infimum is strictly positive.  $\square$

### 3. MAIN TECHNICAL ESTIMATES

Before we derive our main theorems stated in Section 2 we develop the crucial technical ingredient underlying their proofs in the present section. Throughout this section we shall always assume that  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  fulfill Hypothesis 2.1.

In what follows we pick  $\delta > 0$ ,  $e \geq 0$ , and define spectral projections

$$\overline{\Pi}_\delta := \mathbb{1}_{(e-\delta, e+\delta)}(H_f), \quad \Pi_\delta := \mathbb{1} - \overline{\Pi}_\delta.$$

We shall use the following simple observation: Recall the notation

$$(a(k) \psi)^{(n)}(k_1, \dots, k_n) := (n+1)^{1/2} \psi^{(n+1)}(k, k_1, \dots, k_n),$$

almost everywhere, for  $\psi = \{\psi^{(\ell)}\}_{\ell=0}^\infty \in \mathcal{F}_b$  and  $n \in \mathbb{N}_0$ , and  $a(k)\Omega := 0$ . Suppose that  $f_1, \dots, f_n \in \mathfrak{h}$  have supports in  $\{\omega \geq 2\delta\} \times \mathbb{Z}_2$ . By the pull-through formula,  $a(k) F(H_f) = F(H_f + \omega(\mathbf{k})) a(k)$ , we have, for every  $\psi \in \mathcal{F}_b$ ,

$$\begin{aligned} & a(f_1) \dots a(f_n) \overline{\Pi}_\delta \psi \\ &= \int \overline{f_1(k_1)} \dots \overline{f_n(k_n)} \mathbb{1}_{(e-\delta, e+\delta)} \left( H_f + \sum_{j=1}^n \omega(\mathbf{k}_j) \right) a(k_1) \dots a(k_n) \psi dk_1 \dots dk_n, \end{aligned}$$

where  $k_j = (\mathbf{k}_j, \lambda_j)$ ,  $j = 1, \dots, n$ . Of course, if  $\mathbb{1}_{(e-\delta, e+\delta)}(t + \Sigma) \neq 0$  with  $\Sigma \geq 2\delta$ , then  $|t - e| \geq \Sigma - |t + \Sigma - e| \geq \delta$ . That is,

$$a(f_1) \dots a(f_n) \overline{\Pi}_\delta = \Pi_\delta a(f_1) \dots a(f_n) \overline{\Pi}_\delta$$

on  $\mathcal{F}_b$ , or,

$$a(f_1) \dots a(f_n) = a(f_1) \dots a(f_n) \Pi_\delta + \Pi_\delta a(f_1) \dots a(f_n) \overline{\Pi}_\delta$$

on the domain of  $H_f^{n/2}$ . Combining this with (2.2) we obtain in particular

$$\begin{aligned} & |\langle \phi, a(f_1) \dots a(f_n) \psi \rangle| \\ (3.1) \quad & \leq \left( \prod_{j=1}^n \|f_j\|_{1/2} \right) \left( \|\phi\| \|H_f^{n/2} \Pi_\delta \psi\| + (e + \delta)^{n/2} \|\Pi_\delta \phi\| \|\psi\| \right), \end{aligned}$$

for all  $\phi \in \mathcal{F}_b$  and  $\psi \in \mathcal{D}(H_f^{n/2})$ . Let  $d$ ,  $g$ , and  $r$  be given by Hypothesis 2.1.

**Lemma 3.1.** *With the assumptions and notation explained in the previous paragraphs we find a universal constant,  $\mathbf{c} > 1$ , such that, for every normalized  $\psi \in \mathcal{D}(H_f)$ , all  $e \geq 0$ ,  $\delta > 0$  with  $r(2\delta) \leq 1$ ,  $\varepsilon \in (0, 1]$ , and  $\mathbf{p} \in \mathbb{R}^3$ ,*

$$\begin{aligned} & \|w(\mathbf{p}) \psi\|^2 \geq g^2 - r(2\delta) (2 + 9d \langle \psi, H_f \psi \rangle) \\ (3.2) \quad & - \mathbf{c} d^2 (1 + e + \delta) (1 + |\mathbf{p}|) (\varepsilon + \varepsilon^{-1} \|(H_f + 1) \Pi_\delta \psi\|^2). \end{aligned}$$

*Proof.* We set  $\mathbf{G}_< := \mathbb{1}_{\{\omega < 2\delta\}} \mathbf{G}$ ,  $\mathbf{G}_> := \mathbb{1}_{\{\omega \geq 2\delta\}} \mathbf{G}$ , and  $w_<(\mathbf{p}) := \boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{p}_f + \varphi(\mathbf{G}_<))$ , so that

$$w(\mathbf{p})^2 = w_<(\mathbf{p})^2 + w_<(\mathbf{p}) (\boldsymbol{\sigma} \cdot \varphi(\mathbf{G}_>)) + (\boldsymbol{\sigma} \cdot \varphi(\mathbf{G}_>)) w_<(\mathbf{p}) + \varphi(\mathbf{G}_>)^2 \quad \text{on } \mathcal{C}.$$

Using  $w_{<}(\mathbf{p})^2 \geq 0$ ,  $a^\dagger(\mathbf{G}_{>}) \cdot a(\mathbf{G}_{>}) \geq 0$ , and the canonical commutation relations we obtain by a straightforward computation, for  $\psi \in \mathcal{C}$ ,  $\|\psi\| = 1$ ,

$$\begin{aligned} \langle \psi, w(\mathbf{p})^2 \psi \rangle &\geq 4\operatorname{Re} \langle (\mathbf{p} - \mathbf{p}_f) \psi, a(\mathbf{G}_{>}) \psi \rangle - 2\operatorname{Re} \langle \psi, a(\boldsymbol{\mu} \cdot \mathbf{G}_{>}) \psi \rangle \\ &\quad + 2\operatorname{Re} \langle \psi, i\boldsymbol{\sigma} \cdot a(\boldsymbol{\mu} \times \mathbf{G}_{>}) \psi \rangle + 2\operatorname{Re} \langle \psi, a(\mathbf{G}_{>}) a(\mathbf{G}_{>}) \psi \rangle \\ &\quad + 4\operatorname{Re} \langle a(\mathbf{G}_{>}) \psi, a(\mathbf{G}_{<}) \psi \rangle + 4\operatorname{Re} \langle a^\dagger(\mathbf{G}_{<}) \psi, a(\mathbf{G}_{>}) \psi \rangle + \|\mathbf{G}_{>}\|^2. \end{aligned}$$

Next, we apply (2.2) to the two terms containing  $\mathbf{G}_{<}$  in the last line, (2.4) to the term  $\|\mathbf{G}_{>}\|^2$ , and (2.3) and (3.1) to all remaining terms on the RHS of the previous estimate. Proceeding in this way we arrive at

$$\begin{aligned} &\|w(\mathbf{p})^2 \psi\|^2 \\ &\geq g^2 - r(2\delta)^2 - 4d \left( \|(\mathbf{p} - \mathbf{p}_f) \psi\| \|H_f^{1/2} \Pi_\delta \psi\| + (e + \delta)^{1/2} \|(\mathbf{p} - \mathbf{p}_f) \Pi_\delta \psi\| \right) \\ &\quad - 4d r(2\delta) \left( 2\|H_f^{1/2} \psi\|^2 + \|H_f^{1/2} \psi\| \right) - 2d^2 \left( \|H_f \Pi_\delta \psi\| + (e + \delta) \|\Pi_\delta \psi\| \right) \\ &\quad - \mathbf{c} d \left( \|H_f^{1/2} \Pi_\delta \psi\| + (e + \delta)^{1/2} \|\Pi_\delta \psi\| \right). \end{aligned}$$

Finally, we use  $\|\mathbf{p}_f \psi\|/d \leq \|H_f \psi\| \leq \|H_f \Pi_\delta \psi\| + e + \delta$  and  $\|H_f^{1/2} \Pi_\delta \psi\| \leq \|(H_f + 1) \Pi_\delta \psi\|^{1/2}$  as well as the following consequence of Young's inequality,  $t(1 + t^{1/2}) \leq \varepsilon + 2t^2/\varepsilon$ ,  $t \geq 0$ ,  $0 < \varepsilon \leq 1$ , to obtain (3.2), for all normalized  $\psi \in \mathcal{C}$ . An approximation argument extends it to all  $\psi \in \mathcal{D}(H_f)$  of norm 1.  $\square$

In what follows we abbreviate

$$F_\#(\mathbf{p}) := \hat{\tau}_\#(\mathbf{p})^2 + w(\mathbf{p})^2.$$

**Lemma 3.2.** *Let  $\mathbf{p} > 0$ . Then we find  $\mathbf{c}_0 > 0$ , depending only on  $\mathbf{p}$  and the quantities  $d$ ,  $g$ , and  $r$  in Hypothesis 2.1, such that, for all  $|\mathbf{p}| \leq \mathbf{p}$ ,  $\rho \in (0, 1]$ , and all normalized  $\psi$  in the range of the spectral projection  $\mathbb{1}_{(-\infty, E_\#(\mathbf{p})+\rho)}(H_\#(\mathbf{p}))$ ,*

$$(3.3) \quad \langle \psi, F_\#(\mathbf{p}) \psi \rangle \geq \mathbf{c}_0 - \rho^2.$$

*Proof.* We set  $e := E_\#(\mathbf{p})$  and always assume that  $\delta \in (0, 1]$ . By (2.13) we have  $e \leq \epsilon_0 \equiv \epsilon_0(\mathbf{p}, d)$ . By assumption  $\|\hat{\tau}_\#(\mathbf{p}) \psi + (H_f - e) \psi\| \leq \rho$ , whence

$$2\rho^2 + 2\langle \psi, \hat{\tau}_\#(\mathbf{p})^2 \psi \rangle \geq \|(H_f - e) \psi\|^2 \geq \|(H_f - e) \Pi_\delta \psi\|^2.$$

Moreover,  $\langle \psi, H_f \psi \rangle \leq \langle \psi, H_\#(\mathbf{p}) \psi \rangle \leq e + \rho \leq \epsilon_0 + 1$  and we observe that  $\|(H_f + 1) \Pi_\delta \psi\| \leq (2 + e) \|(H_f - e) \Pi_\delta \psi\|/\delta$ . By virtue of Lemma 3.1 we deduce that the inequality

$$\langle \psi, F_\#(\mathbf{p}) \psi \rangle \geq [a - bx]_+ + x - \rho^2$$

is satisfied at the point  $x = \|(H_f - e) \Pi_\delta \psi\|^2/2$ , where  $[t]_+ := \max\{0, t\}$  and

$$\begin{aligned} a &:= g^2 - r(2\delta) (2 + 9d(1 + \epsilon_0)) - \mathbf{c} d^2 (2 + \epsilon_0) (1 + |\mathbf{p}|) \varepsilon, \\ b &:= 2\mathbf{c} d^2 (2 + \epsilon_0)^3 (1 + |\mathbf{p}|) / \delta^2 \varepsilon. \end{aligned}$$

Finally, we fix  $\delta, \varepsilon \in (0, 1]$  such that  $a \geq g^2/2$  and observe that  $b > 1$  and, hence,  $\inf_{x \geq 0} \{[g^2/2 - bx]_+ + x\} \geq g^2/2b$ .  $\square$

In our applications we need bounds similar to (3.3) but with  $F_\sharp(\mathbf{p})$  replaced by some other functions of  $w(\mathbf{p})$ . In order to derive them in Proposition 3.4 below we consider the spectral measures associated with  $w(\mathbf{p})^2$ ,

$$d\mu_\psi(\lambda) := d\langle \psi, E_\lambda(w(\mathbf{p})^2) \psi \rangle, \quad \psi \in \mathbb{C}^2 \otimes \mathcal{F}_b,$$

and write  $F_\sharp(\mathbf{p}) = f(w(\mathbf{p})^2)$  so that  $f(t) = t^2/4 + t$  in the NR case and  $f(t) = 2(t+1 - \sqrt{t+1})$  in the SR case. In the proof of Proposition 3.4 it is crucial that

$$\int_{\{f \geq b\}} f d\mu_\psi \xrightarrow{b \rightarrow \infty} 0,$$

uniformly for all normalized  $\psi$  in the range of  $\mathbb{1}_{(-\infty, E_\sharp(\mathbf{p})+\rho)}(H_\sharp(\mathbf{p}))$ . To verify the uniformity of the above limit we shall apply (see (3.6)) the following higher order estimate:

**Lemma 3.3.** *For every  $\mathbf{p} \in \mathbb{R}^3$ , it holds  $\mathcal{D}(H_\sharp(\mathbf{p})^{n/2}) \subset \mathcal{D}(H_f^{n/2})$  and*

$$\|H_f^{n/2}(H_\sharp(\mathbf{p}) + 1)^{-n/2}\| \leq \mathfrak{c}(d), \quad n \in \{1, \dots, 8\}.$$

*Proof.* According to Theorems 4.2 and 5.2 of [Ma] and (2.3) we have

$$(3.4) \quad \|H_f^{n/2}(\mathbb{H}_\sharp(0) + 1)^{-n/2}\| \leq \mathfrak{c}(d) (\mathbb{E}_\sharp(0) + 1)^{2n-1}.$$

(The same bound with a less explicit RHS has been obtained earlier in [FGS].) Since  $\mathbb{H}_\sharp(0)$  is unitarily equivalent to the direct integral of the fiber Hamiltonians and since the corresponding unitary transformation ( $U_0$  in (2.11)) commutes with  $H_f$  the LHS of (3.4) is equal to  $\sup_{\varepsilon > 0} \text{ess sup}_{\mathbf{p} \in \mathbb{R}^3} N_\varepsilon(\mathbf{p})$  with

$$N_\varepsilon(\mathbf{p}) := \|F_\varepsilon(H_\sharp(\mathbf{p}) + 1)^{-n/2}\|, \quad \mathbf{p} \in \mathbb{R}^3, \quad F_\varepsilon := H_f^{n/2}(\mathbb{1} + \varepsilon H_f)^{-n/2}.$$

Since  $\mathbf{p} \mapsto (H_\sharp(\mathbf{p}) + 1)^{-1}$  is norm-continuous (see [KöMa2] for the SR case) we know that  $N_\varepsilon$  is continuous on  $\mathbb{R}^3$  and, in particular, its essential supremum is actually a supremum. Furthermore,  $\mathbb{E}_\sharp(0) \leq \mathfrak{c}'(d)$  by Lemma 2.4.  $\square$

**Proposition 3.4.** *Let  $\mathbf{p} > 0$ . Then there exist  $\mathfrak{c}_0, \mathfrak{c}_1 > 0$ , depending only on  $\mathbf{p}$  and the quantities  $d, g$ , and  $r$  in Hypothesis 2.1, such that, for all  $|\mathbf{p}| \leq \mathbf{p}$ ,  $\rho \in (0, 1]$ , and all normalized  $\psi$  in the range of the spectral projection  $\mathbb{1}_{(-\infty, E_\sharp(\mathbf{p})+\rho)}(H_\sharp(\mathbf{p}))$ ,*

$$(3.5) \quad \mu_\psi([a, b]) \geq (\mathfrak{c}_0 - \mathfrak{c}_1/b_0 - 2a - \rho^2)/b_0, \quad 0 < a < b_0 \leq b, \quad a \leq 1.$$

*Proof.* On account of Lemma 3.3 we have, for every  $b > 0$ ,

$$(3.6) \quad \begin{aligned} \int_{\{f \geq b\}} f d\mu_\psi &\leq b^{-1} \int_{\mathbb{R}} f^2 d\mu_\psi \leq \mathfrak{c} b^{-1} \|(w(\mathbf{p})^4 + i) \psi\|^2 \\ &\leq \mathfrak{c}'(\mathbf{p}, d) b^{-1} \|(H_f + 1)^4 \psi\|^2 \leq \mathfrak{c}_1(\mathbf{p}, d) b^{-1}. \end{aligned}$$



Dropping the argument  $(\mathbf{p}, d)$  of  $\mathbf{c}_1$  we infer from (3.3) that, for all  $0 < a < b$ ,

$$\mathbf{c}_0 - \rho^2 \leq \int_{\mathbb{R}} f d\mu_\psi \leq a + b \mu_\psi(\{a \leq f \leq b\}) + \mathbf{c}_1/b,$$

that is,

$$\mu_\psi(\{2a \leq f \leq b\}) \geq (\mathbf{c}_0 - \mathbf{c}_1/b - 2a - \rho^2)/b.$$

Recall that  $\mu_\psi((-\infty, 0)) = 0$ . In the NR case  $t \leq f(t) \leq t(t+1)$  and in the SR case  $t \leq f(t) \leq 2t$  and we readily conclude the proof.  $\square$

**Corollary 3.5.** *Let  $\mathbf{p} > 0$ . Then there exists  $\mathbf{c} > 0$ , depending only on  $\mathbf{p}$  and the quantities  $d$ ,  $g$ , and  $r$  in Hypothesis 2.1, such that, for all  $|\mathbf{p}| \leq \mathbf{p}$ ,*

$$(3.7) \quad E_\#(\mathbf{p}) \geq \mathbf{c},$$

and, with  $\mathcal{M}_\rho$  denoting the set of all  $\psi$  as in the statement of Proposition 3.4,

$$(3.8) \quad \liminf_{\rho \downarrow 0} \inf_{\psi \in \mathcal{M}_\rho} \langle \psi, \hat{\tau}_\#(\mathbf{p}) (\hat{\tau}_\#(\mathbf{p}) + 1)^{-1} \psi \rangle \geq \mathbf{c}.$$

*Proof.* Fix  $a, b, \rho > 0$  such that the RHS in (3.5) is  $\geq \mathbf{c}_0/2b$  and set  $g(t) := t/2$  in the NR case and  $g(t) = \sqrt{t+1} - 1$  in the SR case. Then

$$\begin{aligned} E_\#(\mathbf{p}) &= \inf_{\psi \in \mathcal{M}_\rho} \langle \psi, H_\#(\mathbf{p}) \psi \rangle \geq \inf_{\psi \in \mathcal{M}_\rho} \langle \psi, \hat{\tau}_\#(\mathbf{p}) \psi \rangle \\ &= \inf_{\psi \in \mathcal{M}_\rho} \int_{\mathbb{R}} g(t) d\mu_\psi(t) \geq g(a) \inf_{\psi \in \mathcal{M}_\rho} \mu_\psi([a, b]) \geq \mathbf{c}_0 g(a)/2b. \end{aligned}$$

The same argument with  $g$  replaced by  $g/(g+1)$  gives (3.8).  $\square$

#### 4. THE NON-RELATIVISTIC CASE

In this section we prove Theorem 2.5 in the NR case,  $\# = \text{nr}$ , by a variational argument employing a trial function (see (4.4)) resembling the one in [CVV]. Thanks to the results of Section 3 we may construct this trial function by means of minimizing sequences for certain fiber Hamiltonians  $H_{\text{nr}}(\mathbf{q})$ ; we do not assume existence of minimizers of  $H_{\text{nr}}(\mathbf{0})$  as in [CVV]. Some further modifications (the unitaries  $U_{\mathbf{q}}$  and  $U_R$  below) allow us to drop the radial symmetry of the external potential assumed earlier. (This restriction has been overcome in [BLV], too.) By the use of  $U_{\mathbf{q}}$  and  $U_R$  it is also immaterial whether  $E_{\text{nr}}$  attains its minimum at zero or not. Finally, we remark that the momentum cut-off  $\chi_\varrho$  in Part (b) of the proof is inserted in order to handle all  $V_- \in L^{3/2}(\mathbb{R}^3)$ . The crucial point is that we cannot expect the second derivatives of the eigenfunctions  $\psi_\lambda$  of  $h_{\text{nr}}(V_\lambda)$ ,  $\lambda > 1$ , to belong to  $L^2(\mathbb{R}^3)$ , if the singularities of  $V_-$  are not square-integrable. Therefore, we have to regularize the expression  $\partial_{x_\nu} \psi_\lambda$  in momentum space before we take further derivatives of it in some of the estimates below. What is important to observe the enhanced binding is that the norm of  $\chi_\varrho \partial_{x_\nu} \psi_\lambda$  does not vanish in the limit  $\lambda \downarrow 1$  after suitable normalization of  $\psi_\lambda$ . This follows from Theorem A.2 whose proof is inspired by [SøSt].

*Proof of Theorem 2.5: the NR case.* Let  $\mathbf{p}_*$  be as in Lemma 2.4 and  $\mathbf{p}_* \in \mathbb{R}^3$ ,  $|\mathbf{p}_*| \leq \mathbf{p}_*$ . Let  $R \in \text{SO}(3, \mathbb{R})$  be some rotation matrix to be specified later on. Setting  $\mathbf{q} := R\mathbf{p}_*$  and  $\mathbf{v}(\mathbf{p}) := \mathbf{p} - \mathbf{p}_f + \varphi(\mathbf{G})$ ,  $\mathbf{p} \in \mathbb{R}^3$ , we have the following identity for the fiber Hamiltonians,

$$(4.1) \quad H_{\text{nr}}(\mathbf{q} + \mathbf{p}) = H_{\text{nr}}(\mathbf{q}) + \mathbf{p} \cdot \mathbf{v}(\mathbf{q}) + \frac{\mathbf{p}^2}{2}, \quad \mathbf{p} \in \mathbb{R}^3,$$

in the sense of quadratic forms on, e.g., the domain  $\mathcal{D}(H_f) \supset \mathcal{D}(H_{\text{nr}}(\mathbf{p}))$ ,  $\mathbf{p} \in \mathbb{R}^3$ . Since we assume that  $(\omega, \boldsymbol{\mu}, \mathbf{G})$  have the special form of Example 2.2(i) there is a unitary operator,  $U_R \in \mathcal{B}(\mathbb{C}^2 \otimes \mathcal{F}_b)$ , depending on  $R$  and the choice of the polarization vectors, such that

$$(4.2) \quad U_R H_{\text{nr}}(\mathbf{q}) U_R^* = H_{\text{nr}}(\mathbf{p}_*), \quad U_R \mathbf{p} \cdot \mathbf{v}(\mathbf{q}) U_R^* = \mathbf{p} \cdot (R \mathbf{v}(\mathbf{p}_*));$$

see, e.g., [Hi2, Lemma 2.10] for details. Next, we pick normalized

$$\phi_j \in \text{Ran}(\mathbb{1}_{(-\infty, E_{\text{nr}}(\mathbf{p}_*)+1/j)}(H_{\text{nr}}(\mathbf{p}_*))) , \quad j \in \mathbb{N} ,$$

so that  $\phi_j \in \mathcal{D}(H_f^2)$  by Lemma 3.3. Furthermore, we pick normalized  $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^3)$  satisfying  $\psi_1 = \bar{\psi}_1$ ,  $\psi_2 = -\bar{\psi}_2$ , and let  $\{\phi'_j\}_j$  denote another sequence of normalized vectors in  $\mathcal{D}(H_f^2) \subset \mathbb{C}^2 \otimes \mathcal{F}_b$  to be specified later on satisfying

$$(4.3) \quad \langle \phi_j, \phi'_j \rangle \in \mathbb{R} .$$

For  $\eta < 0$ , we finally define unnormalized trial vectors  $\psi_{\text{tr},j}$  by

$$(4.4) \quad \psi_{\text{tr},j} := U_{\mathbf{q}}^* \mathcal{F}^* \hat{\psi}_{\text{tr},j}, \quad \hat{\psi}_{\text{tr},j}(\mathbf{p}) := \hat{\psi}_1(\mathbf{p}) U_R^* \phi_j + \eta \hat{\psi}_2(\mathbf{p}) U_R^* \phi'_j ,$$

where  $U_{\mathbf{q}}$  is the unitary operator defined in (2.11). By definition,

$$(4.5) \quad \text{Re} \langle \psi_1, \psi_2 \rangle = 0, \quad \langle \psi_1, \nabla_{\mathbf{x}} \psi_1 \rangle = \langle \psi_2, \nabla_{\mathbf{x}} \psi_2 \rangle = 0 ,$$

$$(4.6) \quad \text{Re} \langle \psi_1, h_{\text{nr}}(V) \psi_2 \rangle = 0, \quad \text{Im} \langle \psi_1, -i \nabla_{\mathbf{x}} \psi_2 \rangle = 0 ,$$

$$(4.7) \quad \|\psi_{\text{tr},j}\|^2 = 1 + \eta^2 .$$

In view of (2.12) and (4.7) we have

$$\begin{aligned} (1 + \eta^2) \mathbb{E}_{\text{nr}}(V) &\leq \langle \psi_{\text{tr},j}, \mathbb{H}_{\text{nr}}(V) \psi_{\text{tr},j} \rangle \\ &= \int_{\mathbb{R}^3} \langle \hat{\psi}_{\text{tr},j}(\mathbf{p}), H_{\text{nr}}(\mathbf{q} + \mathbf{p}) \hat{\psi}_{\text{tr},j}(\mathbf{p}) \rangle_{\mathbb{C}^2 \otimes \mathcal{F}_b} d^3 \mathbf{p} + \langle \psi_{\text{tr},j}, V \psi_{\text{tr},j} \rangle . \end{aligned}$$

Employing (4.1)–(4.6) we find after some easy computations

$$\begin{aligned} \mathbb{E}_{\text{nr}}(V) - (c_V + \mathbf{c} d^2) \eta^2 &\leq \langle \psi_1, h_{\text{nr}}(V) \psi_1 \rangle + \langle \phi_j, H_{\text{nr}}(\mathbf{p}_*) \phi_j \rangle \\ &\quad + \eta^2 \langle \psi_2, h_{\text{nr}}(V) \psi_2 \rangle + \eta^2 \langle \phi'_j, H_{\text{nr}}(\mathbf{p}_*) \phi'_j \rangle \\ &\quad + 2\eta \text{Re} \{ \langle \psi_1, \psi_2 \rangle \langle \phi_j, H_{\text{nr}}(\mathbf{p}_*) \phi'_j \rangle \} \\ (4.8) \quad &\quad + 2\eta \langle \psi_1, -i \nabla_{\mathbf{x}} \psi_2 \rangle \cdot \text{Re} \langle R \mathbf{v}(\mathbf{p}_*) \phi_j, \phi'_j \rangle . \end{aligned}$$

In the first line we also applied the lower bound (2.10). By virtue of Corollary 3.5 we find some  $\mathbf{c}_0 > 0$ , depending only on  $\mathbf{p}_* \equiv \mathbf{p}_*(d)$  and the quantities  $d$ ,  $g$ , and  $r$  in Hypothesis 2.1, such that

$$(4.9) \quad (3/2) \liminf_{j \rightarrow \infty} \langle \phi_j, \mathbf{v}(\mathbf{p}_*)^2 \phi_j \rangle \geq \liminf_{j \rightarrow \infty} \langle \phi_j, \hat{\tau}_{\text{nr}}(\mathbf{p}_*) \phi_j \rangle \geq \mathbf{c}_0.$$

By the higher order estimates of Lemma 3.3 we know that  $\sup_j \|v_\nu(\mathbf{p}_*) \phi_j\|$  is finite, where  $v_\nu(\mathbf{p}_*)$  is the  $\nu$ -th component of  $\mathbf{v}(\mathbf{p}_*)$ . Passing to a suitable subsequence, if necessary, we may thus define

$$c_1(\nu) := \lim_{j \rightarrow \infty} \|v_\nu(\mathbf{p}_*) \phi_j\|, \quad \nu = 1, 2, 3, \quad c_1^2 := \frac{1}{3} \sum_{\nu=1}^3 c_1(\nu)^2 \geq 2\mathbf{c}_0/9,$$

For  $\nu_0 \in \{1, 2, 3\}$  with  $c_1(\nu_0)^2 \geq c_1^2$ , we set

$$\phi'_j := v_{\nu_0}(\mathbf{p}_*) \phi_j \cdot \|v_{\nu_0}(\mathbf{p}_*) \phi_j\|^{-1}.$$

This choice is allowed since  $\langle \phi_j, \phi'_j \rangle \in \mathbb{R}$  due to the fact that  $v_{\nu_0}(\mathbf{p}_*)$  is symmetric and  $\phi'_j \in \mathcal{D}(H_f^2)$  by Lemma 3.3 and a straightforward calculation. Furthermore,  $\langle \phi'_j, H_{\text{nr}}(\mathbf{p}_*) \phi'_j \rangle \leq \mathbf{c}'(\mathbf{p}_*, d) \|(H_f + 1)^2 \phi_j\|^2 / c_1(\nu_0)$  and the higher order estimates ensure the existence of some  $c_2 > 0$ , depending only on  $\mathbf{p}_*$  and  $d$ , such that  $\langle \phi'_j, H_{\text{nr}}(\mathbf{p}_*) \phi'_j \rangle \leq c_2$ , for all  $j$ . Since also  $\sup_j |\langle v_\nu(\mathbf{p}_*) \phi_j, \phi'_j \rangle| \leq \mathbf{c}(\mathbf{p}_*, d) \sup_j \|(H_f + 1) \phi_j\|^2 < \infty$  we may define, at least along some suitable subsequence,

$$\boldsymbol{\alpha} := \lim_{j \rightarrow \infty} \text{Re} \langle \mathbf{v}(\mathbf{p}_*) \phi_j, \phi'_j \rangle, \quad \text{so that} \quad |\boldsymbol{\alpha}| \geq c_1.$$

In fact, the  $\nu_0$ -component of  $\boldsymbol{\alpha}$  is just equal to  $c_1(\nu_0)$ . We are still free to choose the rotation  $R$  in (4.8). We set  $\boldsymbol{\beta} := \langle \psi_1, -i\nabla_{\mathbf{x}} \psi_2 \rangle \in \mathbb{R}^3$  and choose it such that  $\boldsymbol{\beta} \cdot (R \boldsymbol{\alpha}) = |\boldsymbol{\alpha}| |\boldsymbol{\beta}| \geq c_1 |\boldsymbol{\beta}|$ . Plugging the new notation into (4.8), passing to the limit  $j \rightarrow \infty$ , and taking also

$$\lim_{j \rightarrow \infty} \langle \phi_j, H_{\text{nr}}(\mathbf{p}_*) \phi'_j \rangle = E_{\text{nr}}(\mathbf{p}_*) \lim_{j \rightarrow \infty} \langle \phi_j, \phi'_j \rangle \in \mathbb{R}$$

into account, we readily arrive at

$$(4.10) \quad \begin{aligned} & \mathbb{E}_{\text{nr}}(V) - E_{\text{nr}}(\mathbf{p}_*) - \langle \psi_1, h_{\text{nr}}(V) \psi_1 \rangle \\ & \leq \eta^2 \{ \langle \psi_2, h_{\text{nr}}(V) \psi_2 \rangle + c_2 + c_{\tilde{V}} + \mathbf{c} d^2 \} + 2\eta c_1 |\boldsymbol{\beta}|. \end{aligned}$$

Here  $\tilde{V}$  is any potential satisfying the condition in (2.9), i.e.  $c_{\tilde{V}} < \infty$ , and  $\tilde{V} \leq V$ . In particular, the curly bracket  $\{\dots\}$  in (4.10) is strictly positive. Minimizing the RHS with respect to  $\eta < 0$  and applying (2.13) on the LHS we thus obtain

$$(4.11) \quad \mathbb{E}_{\text{nr}}(V) - \mathbb{E}_{\text{nr}}(0) - \langle \psi_1, h_{\text{nr}}(V) \psi_1 \rangle \leq \frac{-|\boldsymbol{\beta}|^2 / \mathbf{c}_1}{\langle \psi_2, h_{\text{nr}}(V) \psi_2 \rangle + c_{\tilde{V}} + \mathbf{c}_1},$$

where  $\mathbf{c}_1 > 0$  depends only on  $\mathbf{p}_* = \mathbf{p}_*(d)$ ,  $d$ ,  $g$ , and  $r$  and  $\beta = \langle \psi_1, -i\nabla_{\mathbf{x}}\psi_2 \rangle$ . Since  $C_0^\infty(\mathbb{R}^3)$  is a form core for  $h_{\text{nr}}(V)$  an approximation argument shows that (4.11) is actually valid, for every real-valued normalized  $\psi_1 \in \mathcal{Q}(h_{\text{nr}}(V))$ . If  $\mathcal{Q}(h_{\text{nr}}(V)) \subset H^1(\mathbb{R}^3)$ , then (4.11) applies to every purely imaginary normalized  $\psi_2 \in \mathcal{Q}(h_{\text{nr}}(V))$ .

(a): Under the assumptions of Theorem 2.5(a) the electron operator  $h_{\text{nr}}(V)$  has a normalized, real-valued ground state eigenfunction,  $\psi$ . We set  $\psi_1 := \psi$ . Since the distributional Laplacian  $\Delta_{\mathbf{x}}\psi \in \mathcal{D}'(\mathbb{R}^3)$  is non-zero we find some  $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$  such that  $\langle \psi, -\Delta_{\mathbf{x}}\phi \rangle > 0$ . We choose  $\mu \in \{1, 2, 3\}$  such that  $\langle \psi, -\partial_{x_\mu}^2\phi \rangle > 0$  and set  $\psi_2 := -i\partial_{x_\mu}\phi/\|\partial_{x_\mu}\phi\|$ . Then  $\beta \neq 0$  and the assertion follows from (4.11).

(b): We replace  $V$  by  $V_\mu$ , for  $\mu \in (0, 1)$ . Since  $\lambda = 1$  is the coupling constant threshold there is a normalized, strictly positive ground state eigenfunction  $\psi_\lambda > 0$  of  $h_{\text{nr}}(V_\lambda)$ , for all  $\lambda > 1$ , i.e.  $h_{\text{nr}}(V_\lambda)\psi_\lambda = e_\lambda\psi_\lambda$  with  $e_\lambda < 0$ . According to Theorem A.2 we find a sequence  $\lambda_j \downarrow 1$  such that the vectors  $(-\Delta)^{1/2}\psi_{\lambda_j}/\|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|$  converge to some non-zero limit. Passing to a subsequence, if necessary, we may assume that  $\|(-\Delta)^{1/2}\psi_{\lambda_j}\| \leq 3^{1/2}\|\partial_{x_\nu}\psi_{\lambda_j}\|$ , for some fixed  $\nu \in \{1, 2, 3\}$  and all  $j$ . Then the vectors  $\partial_{x_\nu}\psi_{\lambda_j}/\|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|$  also have a non-zero limit. Let  $\chi_\varrho := \mathbb{1}_{(-\Delta)^{1/2} \leq \varrho}$  be a cut-off in momentum space. Then we find  $\alpha, \varrho > 0$  such that  $\|\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j}\| \geq \alpha \|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|$ , for large  $j$ . Now, we choose  $\psi_1 := \psi_{\lambda_j}$  and  $\psi_2 := -i\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j}/\|\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j}\|$ . Notice that  $\psi_2$  is purely imaginary because  $\psi_{\lambda_j}$  is real-valued and the cut-off  $\chi_\varrho$  is symmetric about the origin in momentum space.

Since  $V_\pm \in (L^{3/2} + L^\infty)(\mathbb{R}^3)$  we know that  $\psi_1 \in H^1(\mathbb{R}^3)$  and we may write  $\beta$  as  $\beta = \langle -i\nabla_{\mathbf{x}}\psi_{\lambda_j}, -i\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j} \rangle / \|\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j}\|$ , which shows that

$$|\beta| \geq \|\chi_\varrho \partial_{x_\nu}\psi_{\lambda_j}\| \geq \alpha \|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|.$$

Furthermore, we choose  $\tilde{V} := -2V_-$  in (4.11) for all  $\lambda \leq 2$ . Applying (4.11) with  $V = V_\mu$ , taking the above remarks into account, and using  $h_{\text{nr}}(V_\mu) = h_{\text{nr}}(V_\lambda) + (\lambda - \mu)V_-$  we arrive at

$$\mathbb{E}_{\text{nr}}(V_\mu) - \mathbb{E}_{\text{nr}}(0) - e_{\lambda_j} - (\lambda_j - \mu)\langle \psi_{\lambda_j}, V_- \psi_{\lambda_j} \rangle \leq \frac{-\alpha^2 \|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|^2 / \mathbf{c}_1}{\langle \psi_2, h_{\text{nr}}(V_\mu) \psi_2 \rangle + c_{-2V_-} + \mathbf{c}_1},$$

for  $0 < \mu < 1 < \lambda_j \leq 2$ , where

$$\langle \psi_2, h_{\text{nr}}(V_\mu) \psi_2 \rangle \leq \varrho^2/2 + \|V_{+,2}\|_\infty + \mathbf{c} \|V_{+,1}\|_{3/2} \varrho^2 =: \mathbf{c}(V_\pm),$$

since  $\text{supp}(\hat{\psi}_2) \subset \{|\mathbf{p}| \leq \varrho\}$  and  $\langle \psi_2, V_{+,1}\psi_2 \rangle \leq \mathbf{c} \|V_{+,1}\|_{3/2}^2 \|\nabla\psi_2\|^2$  by Hölder's and Sobolev's inequalities. Hence,

$$\frac{\mathbb{E}_{\text{nr}}(V_\mu) - \mathbb{E}_{\text{nr}}(0)}{\|(\lambda_j V_-)^{1/2}\psi_{\lambda_j}\|^2} - \frac{\lambda_j - \mu}{\lambda_j} \leq \frac{-\alpha^2 / \mathbf{c}_1}{\mathbf{c}(V_\pm) + c_{-2V_-} + \mathbf{c}_1} =: -\mathbf{c}_*.$$

Now, fix  $\delta > 0$  such that  $2\delta = \mathbf{c}_\star \equiv \mathbf{c}_\star(d, g, r, V_\pm)$ , and then fix  $j_0$  such that  $(\lambda_{j_0} - 1 + \delta)/\lambda_{j_0} - \mathbf{c}_\star \leq -\mathbf{c}_\star/4$ . Then  $\|(\lambda_{j_0} V_-)^{1/2} \psi_{\lambda_{j_0}}\|$  is some  $(d, g, r, V_\pm)$ -dependent constant and we conclude.  $\square$

## 5. THE SEMI-RELATIVISTIC CASE

In this section we prove the statements of Theorem 2.5 in the SR case. We also derive the bound (2.19) asserted in Theorem 2.9.

Recall the definitions of  $S$  and  $T_{\text{sr}}$  in (2.18) and  $\mathbf{v}(\mathbf{p}) = \mathbf{p} - \mathbf{p}_f + \varphi(\mathbf{G})$ . We have  $S = T_{\text{sr}}/(T_{\text{sr}} + 1)$ . Given  $\mathbf{p}, \mathbf{p}_* \in \mathbb{R}^3$ , we shall use the following notation for resolvents, where  $\eta > 0$ ,

$$R_1(\eta) := (w(\mathbf{p}_*)^2 + \mathbf{p}^2 + 1 + \eta)^{-1}, \quad R_2(\eta) := (w(\mathbf{p}_* + \mathbf{p})^2 + 1 + \eta)^{-1}.$$

**Lemma 5.1.** (a) For all  $\phi \in \mathcal{D}(H_f)$ ,  $\|\phi\| = 1$ , and  $\mathbf{p}, \mathbf{p}_* \in \mathbb{R}^3$ , we have

$$(5.1) \quad \begin{aligned} \langle \phi, H_{\text{sr}}(\mathbf{p}_* + \mathbf{p}) \phi \rangle &\leq \langle \phi, H_{\text{sr}}(\mathbf{p}_*) \phi \rangle - S(\mathbf{p}) \langle \phi, \hat{\tau}_{\text{sr}}(\mathbf{p}_*) (\hat{\tau}_{\text{sr}}(\mathbf{p}_*) + 1)^{-1} \phi \rangle \\ &\quad + T_{\text{sr}}(\mathbf{p}) + \int_0^\infty 2 \langle R_1(\eta) \phi, \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi \rangle \frac{\eta^{1/2} d\eta}{\pi}. \end{aligned}$$

(b) The bound (2.19) holds true.

*Proof.* The following proof of (a) is a strengthened version of an argument used to derive a non-strict inequality on the binding energy in [KMS1].

As a consequence of Lemma A.1 of [KöMa2] both resolvents  $R_1(\eta)$  and  $R_2(\eta)$  map  $\mathcal{D}(H_f^\nu)$  into itself, for every  $\nu \geq 1/2$ . Taking this into account and writing  $w(\mathbf{p}_* + \mathbf{p})^2 \phi = (w(\mathbf{p}_*)^2 + \mathbf{p}^2) \phi + 2\mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) \phi$ , for  $\phi \in \mathcal{D}(H_f^2)$ , we readily obtain

$$\begin{aligned} \langle \phi, R_1(\eta) \phi \rangle &= \langle \phi, R_2(\eta) \phi \rangle + 2 \langle R_2(\eta) \phi, \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi \rangle \\ &= \langle \phi, R_2(\eta) \phi \rangle + 2 \langle R_1(\eta) \phi, \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi \rangle \\ &\quad - 4 \langle R_2(\eta) \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi, \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi \rangle. \end{aligned}$$

Note that the expression in the last line is negative since  $R_2(\eta)$  is positive. Dropping this term and using the formula

$$A^{1/2} \phi = \int_0^\infty (1 - \eta(A + \eta)^{-1}) \phi \frac{d\eta}{\pi \eta^{1/2}},$$

valid for any positive operator  $A$  in some Hilbert space and  $\phi \in \mathcal{D}(A)$ , we obtain, for normalized  $\phi$ ,

$$(5.2) \quad \begin{aligned} \langle \phi, \hat{\tau}_{\text{sr}}(\mathbf{p}_* + \mathbf{p}) \phi \rangle &\leq \langle \phi, (w(\mathbf{p}_*)^2 + \mathbf{p}^2 + 1)^{1/2} \phi \rangle - 1 \\ &\quad + \int_0^\infty 2 \langle R_1(\eta) \phi, \mathbf{p} \cdot \mathbf{v}(\mathbf{p}_*) R_1(\eta) \phi \rangle \frac{\eta^{1/2} d\eta}{\pi}, \end{aligned}$$

which makes sense since  $\|H_f R_1(\eta) (H_f + 1)^{-1}\| \leq \mathbf{c}(d) (1 + \eta)^{-1}$  and, hence,

$$(5.3) \quad \|\mathbf{v}(\mathbf{p}_*) R_1(\eta) \psi\| \leq \mathbf{c}(\mathbf{p}_*, d) (1 + \eta)^{-1} \|(H_f + 1) \psi\|, \quad \psi \in \mathcal{D}(H_f),$$

by Lemma A.1 in [KöMa2]. Next, we observe that

$$w(\mathbf{p}_*)^2 + \mathbf{p}^2 + 1 = a^2 - 2b, \quad a := \hat{\tau}_{\text{sr}}(\mathbf{p}_*) + T_{\text{sr}}(\mathbf{p}) + 1, \quad b := \hat{\tau}_{\text{sr}}(\mathbf{p}_*) T_{\text{sr}}(\mathbf{p}).$$

In a spectral representation of  $w(\mathbf{p}_*)$  we may now apply the inequality between geometric and arithmetic means,  $\sqrt{a(a - 2b/a)} \leq a - b/a$ , to see that the terms in the first line of the RHS of (5.2), where  $\|\phi\| = 1$ , are not greater than

$$\langle \phi, \hat{\tau}_{\text{sr}}(\mathbf{p}_*) \phi \rangle + T_{\text{sr}}(\mathbf{p}) - T_{\text{sr}}(\mathbf{p}) \langle \phi, \hat{\tau}_{\text{sr}}(\mathbf{p}_*) (\hat{\tau}_{\text{sr}}(\mathbf{p}_*) + T_{\text{sr}}(\mathbf{p}) + 1)^{-1} \phi \rangle,$$

where  $(\hat{\tau}_{\text{sr}}(\mathbf{p}_*) + T_{\text{sr}}(\mathbf{p}) + 1)^{-1} \geq (T_{\text{sr}}(\mathbf{p}) + 1)^{-1}(\hat{\tau}_{\text{sr}}(\mathbf{p}_*) + 1)^{-1}$ . Adding  $\langle \phi, H_{\text{f}} \phi \rangle$  on both sides of (5.2) and employing these bounds we arrive at (5.1) with  $\phi \in \mathcal{D}(H_{\text{f}}^2)$ . Since, for every  $\mathbf{q} \in \mathbb{R}^3$ , we know that  $\mathcal{D}(H_{\text{sr}}(\mathbf{q})) = \mathcal{D}(H_{\text{f}})$  and the graph norms of  $H_{\text{sr}}(\mathbf{q})$  and  $H_{\text{f}}$  are equivalent [KöMa2] we obtain (5.1) with  $\phi \in \mathcal{D}(H_{\text{f}})$  by an approximation argument using (5.3).

(b): The integral in the second line of (5.1) is an odd function of  $\mathbf{p}$  and cancels out when we add a copy of (5.1) with  $\mathbf{p}$  replaced by  $-\mathbf{p}$  to it. Therefore, (2.19) follows from (5.1) upon using  $E_{\text{sr}}(\mathbf{p}_* \pm \mathbf{p}) \leq H_{\text{sr}}(\mathbf{p}_* \pm \mathbf{p})$ , inserting normalized vectors  $\phi_j$  in the range of the spectral projection of  $H_{\text{sr}}(\mathbf{p}_*)$  corresponding to the interval  $(-\infty, E_{\text{sr}}(\mathbf{p}_*) + 1/j]$  and applying (3.8).  $\square$

*Proof of Theorem 2.5: The SR case.* Let  $\mathbf{p}_* > 0$  be the parameter appearing in Lemma 2.4 and set  $\mathbf{q} := \mathbf{p}_*$  in (2.12), where  $|\mathbf{p}_*| \leq \mathbf{p}_*$ . We apply (5.1) to estimate the expectation of

$$\mathcal{F} U_{\mathbf{p}_*} \mathbb{H}_{\text{sr}}(V) U_{\mathbf{p}_*}^* \mathcal{F}^* = \int_{\mathbb{R}^3}^{\oplus} H_{\text{sr}}(\mathbf{p}_* + \mathbf{p}) d^3 \mathbf{p} + \mathcal{F} V \mathcal{F}^*$$

in a trial vector  $\hat{\psi}_{\text{tr}}(\mathbf{p}) = \hat{\psi}_1(\mathbf{p}) \phi$  with  $\phi \in \mathcal{D}(H_{\text{f}}^2)$ ,  $\|\phi\| = 1$ , and  $\psi_1 \in C_0^\infty(\mathbb{R}^3)$ ,  $\bar{\psi}_1 = \psi_1$ , i.e.  $|\hat{\psi}_1(\mathbf{p})| = |\hat{\psi}_1(-\mathbf{p})|$ . Since the integral in the second line of (5.1) is an odd function of  $\mathbf{p}$  it drops out when we integrate with respect to the symmetric measure  $|\hat{\psi}_1(\mathbf{p})|^2 d^3 \mathbf{p}$  and we arrive at

$$\begin{aligned} \mathbb{E}_{\text{sr}}(V) \|\psi_1\|^2 &\leq \int_{\mathbb{R}^3} |\hat{\psi}_1(\mathbf{p})|^2 \langle \phi, H_{\text{sr}}(\mathbf{p}_* + \mathbf{p}) \phi \rangle d^3 \mathbf{p} + \langle \psi_1, V \psi_1 \rangle \\ &\leq \langle \psi_1, h_{\text{sr}}(V) \psi_1 \rangle + \|\psi_1\|^2 \langle \phi, H_{\text{sr}}(\mathbf{p}_*) \phi \rangle \\ (5.4) \quad &\quad - \int_{\mathbb{R}^3} |\hat{\psi}_1(\mathbf{p})|^2 S(\mathbf{p}) d^3 \mathbf{p} \langle \phi, \hat{\tau}_{\text{sr}}(\mathbf{p}_*) (\hat{\tau}_{\text{sr}}(\mathbf{p}_*) + 1)^{-1} \phi \rangle. \end{aligned}$$

Let  $\phi_j$  be as in the proof of Lemma 5.1(b) so that  $\langle \phi_j, H_{\text{sr}}(\mathbf{p}_*) \phi_j \rangle \rightarrow E_{\text{sr}}(\mathbf{p}_*)$ . Substituting  $\phi_j$  for  $\phi$  in (5.4), passing to the limit  $j \rightarrow \infty$ , and taking (3.8) into account we deduce that

$$(\mathbb{E}_{\text{sr}}(V) - E_{\text{sr}}(\mathbf{p}_*)) \|\psi_1\|^2 \leq \langle \psi_1, h_{\text{sr}}(V) \psi_1 \rangle - \mathfrak{c} \langle \psi_1, S(\hat{\mathbf{p}}) \psi_1 \rangle, \quad \hat{\mathbf{p}} := -i \nabla_{\mathbf{x}},$$

where  $\mathfrak{c} > 0$  depends only on  $\mathbf{p}_* \equiv \mathbf{p}_*(d)$ ,  $d$ ,  $g$ , and  $r$ . Applying (2.13) yields

$$(5.5) \quad (\mathbb{E}_{\text{sr}}(V) - \mathbb{E}_{\text{sr}}(0)) \|\psi_1\|^2 \leq \langle \psi_1, h_{\text{sr}}(V) \psi_1 \rangle - \mathfrak{c} \langle \psi_1, S(\hat{\mathbf{p}}) \psi_1 \rangle.$$

Since  $C_0^\infty(\mathbb{R}^3)$  is a form core for  $h_{\text{sr}}(V)$  the previous bound actually holds true, for every real-valued normalized  $\psi_1 \in \mathcal{Q}(h_{\text{sr}}(V))$ .

(a): First, we prove the increase of binding energy. Under the conditions of Theorem 2.5(a) we may choose  $\psi_1$  to be a real-valued, normalized ground state eigenfunction of  $h_{\text{sr}}(V)$  corresponding to the ground state energy  $e_{\text{sr}}(V) < 0$ . Then  $\langle \psi_1, S(\hat{\mathbf{p}}) \psi_1 \rangle > 0$  is a constant depending only on  $V$  and (5.5) yields

$$\mathbb{E}_{\text{sr}}(V) - \mathbb{E}_{\text{sr}}(0) \leq e_{\text{sr}}(V) - \mathfrak{c} \langle \psi_1, S(\hat{\mathbf{p}}) \psi_1 \rangle.$$

(b): Next, we consider enhanced binding abilities. Let the conditions of Theorem 2.5(b) be satisfied so that  $\mu = 1$  is the coupling constant threshold for the family of potentials  $V_\mu = V_+ - \mu V_-$ . For  $\lambda > 1$ , let  $\psi_\lambda$  be the positive eigenvector of  $h_{\text{sr}}(V_\lambda)$  corresponding to the ground state energy  $e_\lambda := e_{\text{sr}}(V_\lambda) < 0$ . We require that the eigenvector of the Birman-Schwinger operator  $K_{\text{sr}}^{V_\lambda}(e_\lambda)$  (defined in (A.2) below) corresponding to  $\psi_\lambda$  is normalized. (Compare Lemma A.1, where we recall the appropriate Birman-Schwinger principle.) For  $0 < \mu \leq 1 < \lambda$ , we then infer from (5.5) with  $\psi_1 := \psi_\lambda$  that

$$(5.6) \quad \begin{aligned} (\mathbb{E}_{\text{sr}}(V_\mu) - \mathbb{E}_{\text{sr}}(0)) \|\psi_\lambda\|^2 &\leq \langle \psi_\lambda, h_{\text{sr}}(V_\mu) \psi_\lambda \rangle - \mathfrak{c} \langle \psi_\lambda, S(\hat{\mathbf{p}}) \psi_\lambda \rangle \\ &\leq e_\lambda \|\psi_\lambda\|^2 - (\mu - \lambda) \langle \psi_\lambda, V_- \psi_\lambda \rangle - \mathfrak{c} \langle \psi_\lambda, S(\hat{\mathbf{p}}) \psi_\lambda \rangle. \end{aligned}$$

Now, by Theorem A.2 there exist  $\lambda_j \downarrow 1$  such that  $\{S(\hat{\mathbf{p}})^{1/2} \psi_{\lambda_j}\}_j$  converges to some non-zero limit and, hence,  $\|S(\hat{\mathbf{p}})^{1/2} \psi_{\lambda_j}\| \geq \alpha$ , for some  $\alpha > 0$  and large  $j$ . Furthermore, the normalization condition imposed on  $\psi_\lambda$  precisely says that  $\|\lambda \langle \psi_\lambda, V_- \psi_\lambda \rangle\| = 1$ ,  $\lambda > 1$ ; see Lemma A.1. Taking these remarks into account we deduce from (5.6) that

$$(5.7) \quad (\mathbb{E}_{\text{sr}}(V_\mu) - \mathbb{E}_{\text{sr}}(0)) \|\psi_{\lambda_j}\|^2 \leq (\lambda_j - \mu)/\lambda_j - \mathfrak{c} \alpha^2,$$

for sufficiently large  $j$ . Now, we conclude as in the NR case.  $\square$

## APPENDIX A. BIRMAN-SCHWINGER OPERATORS AND ZERO-RESONANCES

In this section we consider the electronic one-particle Hamiltonians

$$h_\sharp := h_\sharp(V) = T_\sharp(\hat{\mathbf{p}}) + V, \quad \sharp \in \{\text{nr}, \text{sr}\},$$

acting in the Hilbert space  $L^2(\mathbb{R}^3)$ , with

$$(A.1) \quad T_{\text{nr}}(\hat{\mathbf{p}}) = \frac{1}{2} \hat{\mathbf{p}}^2, \quad T_{\text{sr}}(\hat{\mathbf{p}}) = (\hat{\mathbf{p}}^2 + 1)^{1/2} - 1, \quad \hat{\mathbf{p}} = -i\nabla_{\mathbf{x}},$$

and for a certain class of short range potentials  $V$ . We shall first recall the Birman-Schwinger principle for energies  $e < 0$  (Subsection A.1). After that we discuss the Birman-Schwinger kernels and the existence and some properties of zero-resonances in the singular limit  $e \uparrow 0$  (Subsection A.2).

**A.1. Non-singular Birman-Schwinger kernels.** Let  $\sharp \in \{\text{nr}, \text{sr}\}$  and let  $V_+ \geq 0$  and  $V_- \geq 0$  be the positive and negative parts of  $V = V_+ - V_-$ , respectively. Assume that  $V_\pm \in L^1_{\text{loc}}(\mathbb{R}^3)$  and that  $V_-$  is  $T_\sharp(\hat{\mathbf{p}})$ -form-bounded with relative form bound  $a < 1$ . Then  $T_\sharp(\hat{\mathbf{p}}) + V_+$  and  $T_\sharp(\hat{\mathbf{p}}) + V$  define semi-bounded, closed quadratic forms. Let  $h_\sharp^+$  and  $h_\sharp$  denote the self-adjoint operators representing these forms, respectively. By the KLMN theorem  $\mathcal{Q}(h_\sharp^+) = \mathcal{Q}(h_\sharp) \subset \mathcal{D}(V_-^{1/2})$ . Then we may define a Birman-Schwinger operator,  $K_\sharp^V(e)$ , for every  $e < 0$ , by the formulas

$$(A.2) \quad K_\sharp^V(e) := Y_\sharp(e) Y_\sharp(e)^*, \quad Y_\sharp(e) := V_-^{1/2} (h_\sharp^+ - e)^{-1/2}.$$

In fact,  $Y_\sharp(e)$  is well-defined on  $L^2(\mathbb{R}^3)$  and bounded by the closed graph theorem and  $K_\sharp^V(e) \in \mathcal{B}(L^2(\mathbb{R}^3))$  is self-adjoint. In the next lemma we compare the eigenspaces

$$\begin{aligned} \mathcal{B}_\sharp(e) &:= \{\psi \in L^2(\mathbb{R}^3) : K_\sharp^V(e) \psi = \psi\}, \\ \mathcal{F}_\sharp(e) &:= \{\phi \in \mathcal{D}(h_\sharp) : h_\sharp \phi = e \phi\}. \end{aligned}$$

**Lemma A.1.** *Let  $\sharp \in \{\text{nr}, \text{sr}\}$ . Under the above assumptions on  $V$  and, for every  $e < 0$ , there is a linear bijection  $b \equiv b_\sharp(e) : \mathcal{B}_\sharp(e) \rightarrow \mathcal{F}_\sharp(e)$  satisfying  $\|b\psi\| \leq |e|^{-1/2} \|\psi\|$ , for all  $\psi \in \mathcal{B}_\sharp(e)$ . It is given by*

$$(A.3) \quad b\psi := (h_\sharp^+ - e)^{-1/2} Y_\sharp(e)^* \psi, \quad \psi \in \mathcal{B}_\sharp(e),$$

$$(A.4) \quad b^{-1}\phi = V_-^{1/2} \phi, \quad \phi \in \mathcal{F}_\sharp(e).$$

*Proof.* Assume that  $e < 0$  is an eigenvalue of  $h_\sharp$  and  $\phi$  is a corresponding normalized eigenfunction. For  $\eta \in \mathcal{Q}(h_\sharp)$ , we then get

$$(A.5) \quad \langle (h_\sharp^+ - e)^{1/2} \eta, (h_\sharp^+ - e)^{1/2} \phi \rangle = \langle V_-^{1/2} \eta, V_-^{1/2} \phi \rangle.$$

Now, let  $\eta := (h_\sharp^+ - e)^{-1/2} Y_\sharp(e)^* \eta'$  with some arbitrary  $\eta' \in \mathcal{D}(V_-^{1/2})$  and set  $\psi := V_-^{1/2} \phi \in L^2(\mathbb{R}^3)$ . The condition  $\eta' \in \mathcal{D}(V_-^{1/2})$  ensures that  $Y_\sharp(e)^* \eta' = (h_\sharp^+ - e)^{-1/2} V_-^{1/2} \eta'$ , whence the LHS of (A.5) becomes  $\langle V_-^{1/2} \eta', \phi \rangle = \langle \eta', \psi \rangle$ . We thus obtain  $\langle \eta', \psi \rangle = \langle Y_\sharp(e) Y_\sharp(e)^* \eta', \psi \rangle$  and conclude that  $K_\sharp^V(e) \psi = \psi$ . Suppose  $\psi = 0$ . Then  $0 > e = \langle \phi, h_\sharp^+ \phi \rangle \geq 0$  by (A.5) with  $\eta = \phi$ ; a contradiction!

Conversely, assume that  $\psi \in \mathcal{B}_\sharp(e)$ ,  $\psi \neq 0$ . Defining  $\phi := b\psi \in \mathcal{Q}(h_\sharp)$  as in (A.3) we obtain, for all  $\eta \in \mathcal{Q}(h_\sharp)$ ,

$$\begin{aligned} \langle \eta, (h_\sharp - e)\phi \rangle &= \langle (h_\sharp^+ - e)^{1/2} \eta, Y_\sharp(e)^* \psi \rangle - \langle V_-^{1/2} \eta, V_-^{1/2} \phi \rangle \\ &= \langle V_-^{1/2} \eta, \psi \rangle - \langle V_-^{1/2} \eta, V_-^{1/2} \phi \rangle \\ &= \langle V_-^{1/2} \eta, \psi \rangle - \langle V_-^{1/2} \eta, Y_\sharp(e) Y_\sharp(e)^* \psi \rangle = 0. \end{aligned}$$



We deduce that  $\phi \in \mathcal{D}(h_\#)$  and  $h_\# \phi = e \phi$ . Suppose  $\phi = 0$ . Then  $0 = V_-^{1/2} \phi = K_\#^V(e) \psi = \psi$ , which yields a contradiction.

Finally, we have  $\|Y_\#(e)^* \psi\|^2 = \langle \psi, K_\#(e) \psi \rangle = \|\psi\|^2$  and, hence,  $\|b \psi\| \leq |e|^{-1/2} \|Y_\#(e)^* \psi\| = \|\psi\|$ , for all  $\psi \in \mathcal{B}_\#(e)$ .  $\square$

**A.2. The singular Birman-Schwinger kernel.** In this subsection we study the limit  $\text{s-lim}_{e \uparrow 0} K_\#^V(e)$ . To this end we restrict ourselves to potentials  $V_\pm \in L_{\text{loc}}^1(\mathbb{R}^3)$  with the negative part satisfying  $V_- \in L^{3/2}(\mathbb{R}^3)$  in the non-relativistic case and  $V_- \in L^{3/2} \cap L^3(\mathbb{R}^3)$  in the semi-relativistic case.

Let  $\gamma \in \{1/2, 1\}$ . Since  $|\cdot|^{-\gamma} \in L_w^{3/\gamma}(\mathbb{R}^3)$ , we know that the closure of the densely defined operator  $|\hat{\mathbf{p}}|^{-\gamma} V_-^{1/2}$  is compact, if  $V_-^{1/2} \in L^{3/\gamma}(\mathbb{R}^3)$ ; see [Cw]. In particular, the closure of  $Z_{\text{nr}} := T_{\text{nr}}(\hat{\mathbf{p}})^{-1/2} V_-^{1/2}$  is compact. Moreover,  $T_{\text{sr}}(\hat{\mathbf{p}})^{-1/2} = f_{<}(\hat{\mathbf{p}}) |\hat{\mathbf{p}}|^{-1} + f_{>}(\hat{\mathbf{p}}) |\hat{\mathbf{p}}|^{-1/2}$  where  $f_{<}$  is a bounded function supported in  $\{|\mathbf{p}| \leq 1\}$  and  $f_{>}$  is a bounded function supported in  $\{|\mathbf{p}| \geq 1\}$ . Therefore, the closure of  $Z_{\text{sr}} := T_{\text{sr}}(\hat{\mathbf{p}})^{-1/2} V_-^{1/2}$  is compact, too. In particular, it follows that  $V_-^{1/2}$  is relatively compact with respect to  $T_\#(\hat{\mathbf{p}})^{1/2}$  and, consequently,  $V_-$  is infinitesimally form-bounded with respect to  $T_\#(\hat{\mathbf{p}})$ . (In the NR case the latter assertion also follows from the fact that  $V_-$  is a Rollnik potential.) Therefore, the conclusions of Lemma A.1 are applicable in what follows.

On account of  $T_\#(\hat{\mathbf{p}}) \leq h_\#^+$  and the operator monotonicity of the inversion we have  $\|(h_\#^+ - e)^{-1/2} T_\#(\hat{\mathbf{p}})^{1/2} \psi\| \leq \|\psi\|$ , for all  $\psi \in \mathcal{D}(T_\#(\hat{\mathbf{p}})^{1/2})$  and  $e < 0$ . Using the monotone convergence theorem in a spectral representation of  $h_\#^+$  we infer that  $\text{Ran}(T_\#(\hat{\mathbf{p}})^{1/2}) \subset \mathcal{D}((h_\#^+)^{-1/2})$  and  $\|(h_\#^+)^{-1/2} T_\#(\hat{\mathbf{p}})^{1/2}\| \leq 1$ . Therefore, the densely defined operators  $W_\# := (h_\#^+)^{-1/2} T_\#(\hat{\mathbf{p}})^{1/2}$  and  $X_\# := (h_\#^+)^{-1/2} V_-^{1/2}$  have bounded extensions to the whole Hilbert space and  $\overline{X}_\# = \overline{W}_\# \overline{Z}_\#$  is compact. Furthermore, for  $e < 0$ , it is straightforward to verify that  $Y_\#(e)^* = (h_\#^+)^{1/2} (h_\#^+ - e)^{-1/2} \overline{X}_\#$ . Hence,  $Y_\#(e)^*$  and  $Y_\#(e) = Y_\#(e)^{**} = X_\#^* (h_\#^+)^{1/2} (h_\#^+ - e)^{-1/2}$  are compact and, on account of the spectral calculus and  $\text{Ker}(h_\#^+) = \{0\}$ , their strong limits,

$$Y_\#(0) := \text{s-lim}_{e \uparrow 0} Y_\#(e) = X_\#^*, \quad \text{s-lim}_{e \uparrow 0} Y_\#(e)^* = \overline{X}_\# = Y_\#(0)^*,$$

exist and are compact. By virtue of the uniform boundedness principle we may now define the singular Birman-Schwinger operator

$$(A.6) \quad K_\#^V(0) := Y_\#(0) Y_\#(0)^* = \text{s-lim}_{e \uparrow 0} K_\#^V(e).$$

The next theorem generalizes some results of [SøSt] to a broader class of potentials.

**Theorem A.2.** Let  $\sharp \in \{\text{nr}, \text{sr}\}$ ,  $0 \leq V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$ , and  $0 \leq V_- \in L^{3/2}(\mathbb{R}^3)$ . If  $\sharp = \text{sr}$ , then assume in addition that  $V_- \in L^3(\mathbb{R}^3)$ . Set  $V_\lambda := V_+ - \lambda V_-$  and assume that, for every  $\lambda > 1$ , there is an eigenvalue  $e_\lambda < 0$  of  $h_\sharp(V_\lambda)$  such that  $e_\lambda \rightarrow 0$ ,  $\lambda \downarrow 1$ . Let  $\phi_\lambda$  be a corresponding eigenfunction such that the eigenfunction  $\psi_\lambda = (\lambda V_-)^{1/2} \phi_\lambda$  of the Birman-Schwinger operator has norm 1. Then there is a sequence  $\{\lambda_j\}_j$ ,  $\lambda_j \downarrow 1$ ,  $j \uparrow \infty$ , such that the limits  $\psi := \lim_j \psi_{\lambda_j}$ ,  $\rho := \lim_j (h_\sharp^+)^{1/2} \phi_{\lambda_j}$ , and  $\tilde{\rho} := \lim_j T_\sharp(\hat{\mathbf{p}})^{1/2} \phi_{\lambda_j}$  exist and

$$\psi = K_\sharp^V(0) \psi, \quad \rho = Y_\sharp(0)^* Y_\sharp(0) \rho \neq 0, \quad \tilde{\rho} \neq 0.$$

Moreover, if  $\sharp = \text{nr}$ , then the limit  $\phi := \lim_j \phi_{\lambda_j}$  exists in  $L^6(\mathbb{R}^3)$ . If  $\sharp = \text{sr}$ , then  $\phi_< := \lim_j \mathbb{1}_{|\hat{\mathbf{p}}| \leq 1} \phi_{\lambda_j}$  exists in  $L^6(\mathbb{R}^3)$ ,  $\phi_> := \lim_j \mathbb{1}_{|\hat{\mathbf{p}}| > 1} \phi_{\lambda_j}$  exists in  $L^3(\mathbb{R}^3)$ , and we set  $\phi := \phi_< + \phi_>$ . In both cases  $\phi$  is a zero resonance, i.e. a weak solution of  $h_\sharp \phi \equiv h_\sharp(V_+ - V_-) \phi = 0$  in the sense that

$$(A.7) \quad \int_{\mathbb{R}^3} \phi (T_\sharp(\hat{\mathbf{p}}) \eta + V_+ \eta - V_- \eta) = 0, \quad \eta \in C_0^\infty(\mathbb{R}^3).$$

*Proof.* By Lemma A.1,  $\|\phi_\lambda\| \leq |e_\lambda|^{-1/2} \|\psi_\lambda\| = |e_\lambda|^{-1/2}$  which implies, for all  $\eta \in \mathcal{Q}(h_\sharp^+) \subset \mathcal{D}(V_-^{1/2})$ ,

$$(A.8) \quad \begin{aligned} \langle h_\sharp \eta, \phi_\lambda \rangle &= e_\lambda \langle \eta, \phi_\lambda \rangle + (\lambda - 1) \langle V_-^{1/2} \eta, \psi_\lambda \rangle / \lambda^{1/2} \xrightarrow{\lambda \downarrow 1} 0, \\ \langle h_\sharp^+ \phi_\lambda, \phi_\lambda \rangle &= e_\lambda \|\phi_\lambda\|^2 + \|\psi_\lambda\|^2 / \lambda \leq 1, \quad \lambda > 1. \end{aligned}$$

Therefore, we find a sequence,  $\{\lambda_j\}_j$ ,  $\lambda_j \downarrow 1$ , such that the weak limits  $\psi := \text{w-lim}_j \psi_{\lambda_j}$  and  $\rho := \text{w-lim}_j (h_\sharp^+)^{1/2} \phi_{\lambda_j}$  exist. We have  $\psi = Y_\sharp(0) \rho$  because

$$\langle \eta, \psi \rangle = \lim_j \langle \eta, \psi_{\lambda_j} \rangle = \lim_j \langle V_-^{1/2} \eta, \phi_{\lambda_j} \rangle = \langle X_\sharp \eta, \rho \rangle, \quad \eta \in \mathcal{D}(X_\sharp).$$

In particular,  $\rho = 0$  implies  $\psi = 0$ . Furthermore, since  $Y_\sharp(0)$  is compact we know that  $\{Y_\sharp(0) (h_\sharp^+)^{1/2} \phi_{\lambda_j}\}_j$  contains a strongly convergent subsequence. As it converges weakly to  $Y_\sharp(0) \rho = \psi$  we may assume that  $Y_\sharp(0) (h_\sharp^+)^{1/2} \phi_{\lambda_j} \rightarrow \psi$  strongly, after passing to a subsequence, if necessary. By the Birman-Schwinger principle of Lemma A.1 we know, however, that

$$\phi_{\lambda_j} = (h_\sharp^+ - e_{\lambda_j})^{-1/2} Y_\sharp(e_{\lambda_j})^* \psi_{\lambda_j} = (h_\sharp^+)^{1/2} (h_\sharp^+ - e_{\lambda_j})^{-1} Y_\sharp(0)^* \psi_{\lambda_j}.$$

Since  $Y_\sharp(0)^*$  is compact we may further assume that  $Y_\sharp(0)^* \psi_{\lambda_j} \rightarrow Y_\sharp(0)^* \psi$  strongly, whence

$$(h_\sharp^+)^{1/2} \phi_{\lambda_j} = h_\sharp^+ (h_\sharp^+ - e_{\lambda_j})^{-1} Y_\sharp(0)^* \psi_{\lambda_j} \rightarrow Y_\sharp(0)^* \psi.$$

Hence,  $\rho = \lim_j (h_\sharp^+)^{1/2} \phi_{\lambda_j} = Y_\sharp(0)^* \psi = Y_\sharp(0)^* Y_\sharp(0) \rho$  converges strongly and so does  $\psi = \lim_j Y_\sharp(0) (h_\sharp^+)^{1/2} \phi_{\lambda_j} = Y_\sharp(0) Y_\sharp(0)^* \psi$ . Since also  $Y_\sharp(0) (h_\sharp^+)^{1/2} \eta = X_\sharp^* (h_\sharp^+)^{1/2} \eta = V_-^{1/2} \eta$ , for  $\eta \in \mathcal{Q}(h_\sharp^+) \subset \mathcal{D}(V_-^{1/2})$ , we have  $Y_\sharp(0) (h_\sharp^+)^{1/2} \phi_{\lambda_j} = \lambda_j^{-1/2} \psi_{\lambda_j}$ . It follows that  $\psi_{\lambda_j} \rightarrow \psi$  strongly,  $\|\psi\| = 1$ , and  $\rho \neq 0$ . Furthermore,

it follows that  $T_{\sharp}(\hat{\mathbf{p}})^{1/2} \phi_{\lambda_j} = W_{\sharp}^* (h_{\sharp}^+)^{1/2} \phi_{\lambda_j} \rightarrow W_{\sharp}^* \rho$  strongly, where  $W_{\sharp}^* \rho \neq 0$  since  $W_{\sharp}^*$  is bijective. On account of Sobolev's inequalities for (half-)derivatives this implies the existence of the limits  $\phi$ ,  $\phi_{<}$ ,  $\phi_{>}$  as in the statement. Since, for every  $\eta \in C_0^\infty(\mathbb{R}^3)$ , we have  $V_{\pm} \eta \in L^{6/5} \cap L^{3/2}(\mathbb{R}^3)$  we finally see that (A.7) follows from (A.8).  $\square$

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